Recitation #3 Warm-Up Solutions
1/24/2014

1) At the end of 1994, Walter was half as old as his grandmother. The sum of the years in which they were born is 3838. How old will Walter be at the end of 1999?

Let x be Walter's age in 1994. Then Walter was born in 1994 – x and his grandmother was born in 1994 – 2x. Thus, we have

(1994 – x) + (1994 – 2x) = 3838

3x = 150
x = 50

It follows that Walter will be 55 years old at the end of 1999, since he was 50 at the end of 1994.

2) The student lockers at Olympic High are numbered consecutively beginning with locker number 1. The plastic digits used to number the lockers cost two cents apiece. Thus, it costs 2 cents to label locker number 9 and four cents to label locker number 10. If it costs $137.94 to label all lockers, how many lockers are there at the school?

The first 9 lockers cost 2 cents x 9 = 18 cents.
The next 90 lockers cost 4 cents x 90 = 360 cents.
The next 900 lockers cost 6 cents x 900 = 5400 cents.

Totaling, we get $57.78. It should be fairly clear that the number of lockers is a four digit number at this point. After paying for locker labels 1 – 999, we have $137.94 - $57.78 = $80.16 left to spend. Each of these lockers costs 8 cents, we have 8016/ 8 = 1002 more lockers to put labels on. These lockers are numbered 1000 through 2001. Thus, there are 2001 lockers in the school.

3) Define a sequence of real numbers a_1, a_2, … by $a_1 = 1$ and $a_{n+1}^3 = 99a_n^3$, for all $n \geq 1$. What is a_{100}? (Please answer in the form x^y, where x and y are integers.)

Take the cube root of the given recurrence to get $a_{n+1} = \sqrt[3]{99}a_n$. This defines a geometric sequence with first term 1 and common ratio $\sqrt[3]{99}$. It follows that the 100th term is simply $(\sqrt[3]{99})^{100-1} = 99^{\frac{1}{3}(99)} = 99^{33}$.

4) What is the sum of the digits in the number $2^{1999}5^{2001}$?

$2^{1999}5^{2001} = 2^{1999}5^{1999}5^{2} = (2 \times 5)^{1999}(25)$. Visually, this number is 25 followed by 1999 zeroes. Thus, the sum of all of its digits is 7.

5) Before Ashley started a three hour drive, her car's odometer read 29792, a palindrome. At her destination, her odometer reading was a different palindrome. If Ashley never exceeded 75 miles per hour, what is the greatest possible value of her odometer reading at the end of her trip? What was her greatest possible average speed?

The maximum number of miles Ashley can drive is 3 x 75 = 225, which means our maximum odometer reading would be 29792 + 225 = 30017. First, let's consider palindromes that start with "30" since that is our upper limit. These are of the form 30n03, where n is a digit. Clearly, the only valid digit that fits our constraints is n = 0. Thus, the greatest possible value of her odometer reading is 30003 and her greatest possible average speed was (30003 – 27972)/3 = 70 1/3 miles per hour.

Recitation #3: Set Solutions

1/24/2014

Fill in the blanks for the following groups of statements in questions 1 through 3. As for "showing your work," you should be able to explain in words what the notation in each question means and why your answer is right.

1) $|\{ \emptyset, \{0\}, \{1\}, \{11\}, \{0, 1\}, \{1, 2, 1 + 2\} \}|$ = **_6_**

Two things to point out: first, the null set is counted because it is a set; second, the set elements that have more than one element still only get counted once.

2) $A = \{0, 2, 4, 6\}$
$B = \{0, 1, 2\}$
$C = \mathbf{Z}^{+}$
$(A \cup B) - C$ = **_{0}_**

Note that the answer is not just zero, it is the set containing zero. Also, that's not the same as the empty set.

3) A = {1, 3, 4, 5, 8}, B = { **<u>1, 2, 4, 6</u>** }
A ∪ B = { 1, 2, 3, 4, 5, 6, 8}
A – B = {3, 5, 8}

We obtain B = {1, 2, 4, 6}by noting that 2 and 6 are not already listed in A, but they are in A ∪ B, so they must be in B. Next, we note that 1 and 4 are NOT in A – B. Since they are both in A, they must be in B as well.

4) Prove or disprove the following statement about sets, using any method.

$$A \cap (B \cup C) = (A \cap B) \cup C$$

This statement is false. There are many counterexamples and here is one: A = {0}, B = {0}, C = {1}.

5) Prove or disprove the following statement about finite sets, using any method:

$$\text{If } |A \cap B| = |A \cup B|, \text{ then } A = B.$$

Let's use proof by contradiction. Assume the opposite of what we are trying to prove, namely that A and B are different sets. If this is the case, one of two things is true:

(a) There exists some element x such that $x \in A$ and $x \notin B$.
(b) There exists some element x such that $x \in B$ and $x \notin A$.

In both cases, we find that $x \in A \cup B$ and $x \notin A \cap B$. If both sets are finite, their intersection and union are finite as well. Furthermore, the cardinality of the two sets can't be equal because each element of $A \cap B$ is a part of $A \cup B$, AND there is at least one element, x, which is in $A \cup B$ but not $A \cap B$. Thus, $|A \cup B| \geq |A \cap B| + 1$, implying that $|A \cup B| > |A \cap B|$, which for finite sets means that $|A \cup B| \neq |A \cap B|$. But, this contradicts our assumption. It follows that the initial assumption made was incorrect and A and B are equal sets, as desired. (Note: This statement is not true for infinite sets. If we let A be the integers and B be the negative integers, A – B is the non-negative integers, but rather bizarrely, the cardinality of the non-negative integers is the same as the cardinality of all integers.)