Recitation #2 Warm-Up Solutions
1/17/2014

1) A speaker talked for 60 minutes to a full auditorium. Twenty percent of the audience heard the entire talk and ten percent slept through the entire talk. Half of the remainder of the audience heard one third of the talk and the other heard two thirds of the talk. What was the average number of minutes of the talk heard by the members of the audience?

Solution
Let n people be in the auditorium. Then .2n people hear 60 minutes of the talk, .1n people hear 0 minutes of the talk, .35n people hear 20 minutes of the talk and .35n people hear 40 minutes of the talk. So, to get the average number of minutes the talk was heard, add up the total number of minutes and divide by the number of people listening:

$$\frac{(.2n)60 + (.1n)0 + (.35n)20 + (.35n)40}{n} = \frac{33n}{n} = 33$$

Note that all of the n's in the average cancel out. So, in general, with a problem of this nature, we can solve for the answer by removing all of the n's and multiplying each number of minutes by the fraction of people who heard the talk for that number of minutes, adding across all the people. This is the idea behind a weighted average. Rather than (60+0+20+40)/4, we multiply each number of minutes by how frequent that was in our sample space. So, a more standard calculation to this problem would look like:

$$(.2)60 + (.1)0 + (.35)20 + (.35)40 = \mathbf{33}$$

2) Walter rolls 4 standard six-sided dice and finds that the product of the numbers on the upper faces is 144. What are all the possible sums of the upper four faces?

Solution
Let the four faces be a, b, c and d, with $a \le b \le c \le d$, since order of the dice doesn't matter for this question. We have:

$$abcd = 144 = 2^4 3^2$$

Clearly no more than one die can show a 1. If two did, the maximum product would be 36. First, we solve for solutions were a = 1. In this case, three valid dice rolls that multiply to 144 are 4, 6 and 6. It should be fairly easy to see that no other solutions exist, since 5 can't be in the answer (not in the prime factorization), and if we were to set 4 and 6, the other die is forced to be a 6. Going lower that 4 would force the last die to be higher than 6.

So, one solution is (1, 4, 6, 6) with a sum of 17.

Now, consider all solutions where a = 2. Then we need bcd = 72. Here we generate the following solutions via brute force: (2, 2, 6, 6), and (2, 3, 4, 6). These add up to 16 and 15, respectively.

Now we try a = 3. This yields the solution (3, 3, 4, 4) only. (When we plug in 4 for b, we can't find two numbers that multiply to 12 that are both 4 or greater.) The sum here is 14.

Thus, all possible sums desired are **14, 15, 16 and 17.**

3) Let R be a rectangle. How many circles in the plane of R have a diameter both of whose endpoints are vertices of R?

Solution

Four unique circles are created by choosing the two vertices for each of the four sides. If we choose the circle that has a diameter of one diagonal of the rectangle, by symmetry, that same circle goes through the other two points of the rectangle. So, there are only **5** unique circles that can be drawn by drawing circles with diameters with endpoints at two vertices of R. (There are 6 such choices, but two of them, the pairs of points at both diagonals lead to the same exact circle.)

4) How many different prime numbers are factors of N if

$$log_2\left(log_3\left(log_5(log_7 N)\right)\right) = 11?$$

What are those prime numbers?

Solution

Solve the equation from the outside in:

$$\left(log_3\left(log_5(log_7 N)\right)\right) = 2^{11}, let\ A = 2^{11}$$
$$\left(\left(log_5(log_7 N)\right)\right) = 3^{A}, let\ B = 3^{A}$$
$$log_7 N = 5^{B}, let\ C = 5^{B}$$
$$N = 7^{C}$$

Thus, there is one prime factor of N, it's 7.

5) If $2^{1998} - 2^{1997} - 2^{1996} + 2^{1995} = k(2^{1995})$, what is the value of k?

Solution

$2^3 2^{1995} - 2^2 2^{1995} - 2^1 2^{1995} + 2^{1995} = (2^3 - 2^2 - 2^1 + 1)2^{1995} = 3(2^{1995})$, so **k = 3.**

Recitation #2: Proof Problems
1/17/2014

For questions 1 and 2, use the Rules of Inference and the Law of Contraposition to validate the conclusion drawn below. (Each of the items above the dotted line is a premise, while the conclusion to draw is below the dotted line.) Show each step and state which rule is being used.

1) Prove the desired conclusion using the premises shown below.

$p \vee q$
$\neg r \rightarrow \neg p$
$r \rightarrow s$
$\neg q$
----
s

Solution

Step	Rule
1. $p \vee q$	Premise
2. $\neg q$	Premise
3. p	Disjunctive Syllogism (1, 2)
4. $\neg r \rightarrow \neg p$	Premise
5. $p \rightarrow r$	Contraposition (4)
6. r	Modus ponens (3, 5)
7. $r \rightarrow s$	Premise
8. s	Modus Ponens (6, 7)

2) Prove the desired conclusion using the premises shown below.

$q \rightarrow \neg t$
$p \rightarrow q$
$\neg r \rightarrow (p \wedge q)$
$(p \wedge q) \rightarrow \neg p$

$\neg p \vee (\neg t \wedge r)$

Solution:

1) $p \rightarrow q$	Premise
2) $\neg r \rightarrow (p \wedge q)$	Premise
3) $q \rightarrow \neg t$	Premise
4) $(p \wedge q) \rightarrow \neg p$	Premise
5) $p \rightarrow \neg t$	Law of Syllogism (1, 3)
6) $\neg r \rightarrow \neg p$	Law of Syllogism (2,4)
7) $\neg p \vee \neg t$	Implication identity (5)
8) $r \vee \neg p$	Implication identity (6), double negation
9) $(\neg p \vee \neg t) \wedge (r \vee \neg p)$	Rule of Conjunction (7, 8)
10) $\neg p \vee (\neg t \wedge r)$	Distributive Law (9)

3) Prove that all perfect squares (numbers that can be written as k^2 for some integer k) have an odd number of divisors. Extend the argument to show that all non-perfect squares (all other positive integers) have an even number of divisors. (Hint: Note that most divisors come in pairs. As an example, twelve's divisors are the following pair – (1, 12), (2, 6), (3, 4).)

<u>*Solution*</u>
Consider a perfect square n^2. All of its factors can be placed in one of three groups: factors less than n, the factor equal to n, and the factors greater than n. Clearly there is one factor equal to n. For each unique factor f that is less than n, there is a corresponding factor n^2/f that must be greater than n. (We know that $n^2/n = n$, so if we were to divide n^2 by a number less than n, our result must be greater than n.) Let k equal the number of factors of n^2 strictly less than n. Then, we must have exactly k factors greater than n, since each unique factor less than n matches a unique factor greater than n. (It's a one-to-one mapping.) Counting all the factors we get 2k + 1, which, by definition, is an odd number.

4) At a party, each person counts the number of hands they shook. Prove that if we asked everyone who attended the party after the party was over how many hands they shook and added all of those numbers up, that we would get an even number.

<u>*Solution*</u>
Consider each handshake event. It must occur between two people. Thus, each handshake event adds 2 total to the counts of everyone's handshakes. Thus, if there are H handshakes at a party, the sum of the number of times everyone shook hands must be 2H. Consider the following example for illustration:

Let a party the following attend a party: A, B, C, D, E and F.

Consider the case where the following handshakes occur:

AB, AD, CE, BF

The table below shows the counts of total handshakes for each individual AND the whole group after each handshake:

Handshake#	Handshake	Acount	Bcount	Ccount	Dcount	Ecount	Fcount	Total
1	AB	1	1	0	0	0	0	2
2	AD	2	1	0	1	0	0	4
3	CE	2	1	1	1	1	0	6
4	BF	2	2	1	1	1	1	8

As it can be seen, since each handshake adds to one to two people's count, the column at the very end, the sum of the number of times each person shakes hands must always be even after every handshake. This idea is known as the handshaking lemma. In graph theory, it's the proof that the sum of the degrees of each vertex is always even.

5) The arithmetic mean of two numbers x and y is $\frac{x+y}{2}$ while their geometry mean is $\sqrt{xy}$. Prove that the arithmetic mean of x and y is at least as big as their geometric mean. (Hint: show that the difference of the two is non-negative by rewriting this difference as a perfect square. It will also help to analyze twice this difference and set a = $\sqrt{x}$ and b = $\sqrt{y}$.)

Solution

We aim to prove $\frac{x+y}{2} - \sqrt{xy} \geq 0$.

$$\begin{aligned}\frac{x+y}{2} - \sqrt{xy} &= \frac{1}{2}(x + y - 2\sqrt{xy}) \\ &= \frac{1}{2}\left(x - 2\sqrt{xy} + y\right), let\ a = \sqrt{x}, b = \sqrt{y} \\ &= \frac{1}{2}(a^2 - 2ab + b^2) \\ &= \frac{1}{2}(a-b)^2 \geq 0\end{aligned}$$

Since all perfect squares or real numbers are non-negative.