

Recitation #1: Warm-Up Solutions

1/11/14

1) Consider the sequence 1, -2, 3, -4, ... (More formally, the n^{th} term of the sequence is $(-1)^{n+1}n$.) What is the average of the first 200 terms of the sequence?

Note that the sum of the $(2i-1)^{\text{th}}$ and $2i^{\text{th}}$ for $i \geq 1$ is -1. It follows that the sum of the first 200 terms, or 100 pairs of terms, is -100. The average of these terms must be $-100/200 = -0.5$ ($-1/2$).

2) How many two digit positive integers, n , have the property that the sum of n and the number obtained by reversing the order of its digits is a perfect square? List these numbers.

Let the original number have the digits a and b . Then, $N = 10a + b$ and its reverse is $10b + a$. Adding, we get $(10a + b) + (10b + a) = 11a + 11b = 11(a + b)$. In order for this value to be a perfect square, since 11 is prime $a + b = 11k$, where k is a perfect square. Since a and b are digits $a + b < 19$, thus, our only solutions occur with $k = 1$ and $a + b = 11$. Since a and b are digits, a can range from 2 through 9 and b always equals $11 - a$. This corresponds to the following 8 solutions for N : 29, 38, 47, 56, 65, 74, 83, and 92.

3) The average of 10 positive integers (not necessarily distinct) is 57. If the smallest of these integers is 7, what is the largest possible integer that could be on the list?

If we want to maximize a single number on the list, we should greedily minimize the rest. Given the restriction that 7 is the smallest on the list and that numbers aren't necessarily distinct, set nine of the numbers to 7. We know the sum of the numbers must be $10 \times 57 = 570$, thus the maximum value of the last number is $570 - 9 \times 7 = 570 - 63 = 507$.

4) With proof, determine all possible real values a such that the equation $x^2 + ax + a^2 = 0$ has two distinct real roots.

In order for a quadratic equation to have two distinct real roots, its discriminant (what's under the square root) must be positive. The discriminant of this quadratic is $a^2 - 4(1)a^2 = -3a^2$. Setting up we get the equation $-3a^2 > 0$. Dividing by -3 yields $a^2 < 0$. But, for all real numbers a , this is false. Thus, there are NO values of a for which the original equation has two distinct real roots.

5) Find the two values of x for which the following equation is true: $\log_2(x^2 + 4x) = 5$. (Note: by definition, if $a^b = c$, then $\log_a c = b$.)

By definition of log we get: $2^5 = x^2 + 4x$. We solve for x as follows:

$$\begin{aligned}2^5 &= x^2 + 4x \\x^2 + 4x &= 32 \\x^2 + 4x - 32 &= 0 \\(x + 8)(x - 4) &= 0 \\x &= -8 \text{ or } x = 4.\end{aligned}$$

Recitation #1: Logic Solutions
1/11/14

1) Create a truth table for the following proposition. Include columns for intermediate parts.

$$(p \wedge \bar{q}) \rightarrow (r \vee \bar{p})$$

Answer:

p	q	r	\bar{p}	\bar{q}	$(p \wedge \bar{q})$	$(r \vee \bar{p})$	$(p \wedge \bar{q}) \rightarrow (r \vee \bar{p})$
0	0	0	1	1	0	1	1
0	0	1	1	1	0	1	1
0	1	0	1	0	0	1	1
0	1	1	1	0	0	1	1
1	0	0	0	1	1	0	0
1	0	1	0	1	1	1	1
1	1	0	0	0	0	0	1
1	1	1	0	0	0	1	1

2) Below are the truth tables for NAND, NOR, and XOR (\oplus). Write definitions for these operators in terms of AND, OR, and NOT. Try to explain in words what each one means.

A	B	A NAND B
0	0	1
0	1	1
1	0	1
1	1	0

A	B	A NOR B
0	0	1
0	1	0
1	0	0
1	1	0

A	B	A \oplus B
0	0	0
0	1	1
1	0	1
1	1	0

Answer:

A NAND B == $\neg(A \wedge B)$ == "Not both of A and B."

A NOR B == $\neg(A \vee B)$ == "Neither A nor B."

A XOR B == $(A \vee B) \wedge \neg(A \wedge B)$ == "A or B, but not both."

3) The operator NAND, described above, can be called “computationally complete,” or a “sole sufficient operator.” That means that by itself, it can replace any of the other logical operators. How would you rewrite the statement “ $\neg P$ ” using only NAND?

Answer:

$$\neg P = P \text{ NAND } P$$

Bonus: how to write OR and AND using NAND.

$$P \vee Q = (P \text{ NAND } P) \text{ NAND } (Q \text{ NAND } Q)$$

$$P \wedge Q = (P \text{ NAND } Q) \text{ NAND } (P \text{ NAND } Q)$$

4) Prove the following using the laws of logic and the implication identity:

$$((p \vee \neg p) \wedge (q \vee \neg(\neg q \vee \neg r))) \vee ((p \vee \neg p) \wedge \neg q) \leftrightarrow T$$

Answer:

$$(T \wedge (q \vee \neg(\neg q \vee \neg r))) \vee (T \wedge \neg q) \leftrightarrow$$

$$(T \wedge (q \vee \neg(\neg q \vee \neg r))) \vee \neg q \leftrightarrow$$

$$(T \wedge (q \vee (q \wedge r))) \vee \neg q \leftrightarrow$$

$$q \vee (q \wedge r) \vee \neg q \leftrightarrow$$

$$(q \wedge r) \vee q \vee \neg q \leftrightarrow$$

$$(q \wedge r) \vee (q \vee \neg q) \leftrightarrow$$

$$(q \wedge r) \vee T \leftrightarrow$$

$$T$$

Inverse law $p \vee \neg p \leftrightarrow T$

Identity law $T \wedge \neg q \leftrightarrow \neg q$

DeMorgan's Law

Identity law

Commutative Law

Associative Law

Identity Law

Domination Law

5) Prove the following using the laws of logic and the implication identity:

$$(\neg q \rightarrow \neg p) \rightarrow ((p \wedge q) \vee r) \leftrightarrow r \vee p$$

Answer:

$$(\neg \neg q \vee \neg p) \rightarrow ((p \wedge q) \vee r) \leftrightarrow$$

$$(q \vee \neg p) \rightarrow ((p \wedge q) \vee r) \leftrightarrow$$

$$\neg (q \vee \neg p) \vee ((p \wedge q) \vee r) \leftrightarrow$$

$$(\neg q \wedge p) \vee ((p \wedge q) \vee r) \leftrightarrow$$

$$((\neg q \wedge p) \vee (p \wedge q)) \vee r \leftrightarrow$$

$$((p \wedge (q \vee \neg q)) \vee r) \leftrightarrow$$

$$((p \wedge T) \vee r) \leftrightarrow$$

$$p \vee r \leftrightarrow$$

$$r \vee p$$

Implication identity $p \rightarrow q = \sim p \vee q$

Law of double negation

Implication identity $p \rightarrow q = \sim p \vee q$

DeMorgan's Law and Law of double negation

Associative Law

Distributive Law

Inverse law

Identity Laws

Commutative Laws