Recitation #14 Warm-Up Solutions
4/18/2014

1) Ed and Ann both have lemonade with their lunch. Ed orders the regular size. Ann gets the large lemonade, which is 50% more than the regular. After they consume ¾ of their drinks, Ann gives Ed a third of what she has left, and 2 additional ounces. When they finish their lemonades, they realize that they both drank the same amount. How many ounces of lemonade did they drink together?

Solution

Let the regular sized lemonade be x ounces. Then, Ann's is $\frac{3}{2}x$ ounces. After they consume ¾ of their drinks, Ed has $\frac{x}{4}$ ounces left and Ann has $\frac{3}{8}x$ ounces left, thus a third of what she has is $\frac{1}{8}x$. In sum, Ed drinks his original x ounces plus $\frac{1}{8}x + 2$ more ounces, which equals what Ann drank ($\frac{11}{8}x - 2$ ounces). Setting up the equality, we get:

$$\frac{9}{8}x + 2 = \frac{11}{8}x - 2$$
$$\frac{x}{4} = 4$$
$$x = 16$$

Thus, Ed's cup had 16 ounces and Ann's ahd 24 ounces. Together, they drank **40 ounces.**

2) For how many positive integers *n* is $\frac{n}{30-n}$ also a positive integer?

Solution

Dividing, we find $\frac{n}{30-n} = -1 + \frac{30}{30-n}$. Cycling through the divisors of 30, we find that seven of the eight of them: 1, 2, 3, 5, 6, 10, and 20, form a positive number when plugged into the expression above. *n* = 30 would cause a divide by 0 error and *n* = 0 would make the expression 0, which isn't allowed. Thus, there are **7 positive integers** for which the expression given is also a positive integer.

3) Danica drove her new car on a trip for a whole number of hours, averaging 55 miles per hour. At the beginning of the trip, *abc* miles were displayed on her odometer, where *abc* is a three-digit number with $a \geq 1$ and $a + b + c \leq 7$. At the end of the trip, the odometer showed *cba* miles. What was the odometer reading when the trip was over?

Solution

The trip took $100c + 10b + a - (100a + 10b + c) = 99c - 99a = 99(c - a)$ miles. Setting this equal to 55*n*, where *n* is the number of hours driven, we get $9(c - a) = 5n$. Since gcd(9, 5) = 1, and c and a are digits, we must have $c - a = 5$ and $n = 9$. The only solution that satisfies the given restrictions are a = 1, b = 0 and c = 6. Thus, the desired answer is **601.**

4) A rectangular box has a total surface area of 94 square inches and the sum of the lengths of its edges is 48 inches. What is the sum of the lengths in inches of all of its interior diagonals?

Solution

Let the side lengths of the box by x, y and z. The given information is:

$$2(xy + xz + yz) = 94$$
$$4(x + y + z) = 48$$

Simplifying the second equation we get: $x + y + z = 12$. Squaring this, we find:

$$x^2 + y^2 + z^2 + 2(xy + xz + yz) = 144$$

Since we already know what $2(xy + xz + yz)$ equals, substitute it:

$$x^2 + y^2 + z^2 + 94 = 144$$
$$x^2 + y^2 + z^2 = 50$$

The given expression above is the square of one diagonal of the box. Thus, a single diagonal is $\sqrt{50} = 5\sqrt{2}$. There are four diagonals since there are 8 vertices in a box and each diagonal connects two of them. Thus, the sum of these lengths is $\underline{\mathbf{20\sqrt{2}.}}$

5) Let *P* be a cubic polynomial with P(0) = k, P(1) = 2k and P(-1) = 3k. What is P(2) + P(-2)? (Note: It's impossible to determine P(2) or P(-2) since neither of these values is uniquely determined based on the given information. But, it turns out that P(2) + P(-2) can only take on one value, in terms of k.)

Solution

Let $P(x) = ax^3 + bx^2 + cx + d$. Plugging in P(0), we find that d = k. So now our polynomial is $P(x) = ax^3 + bx^2 + cx + k$. Plugging in the next two given pieces of information, we have:

$$a + b + c + k = 2k$$
$$-a + b - c + k = 3k$$

Subtracting equations, we find $2a + 2c = -k$, so $a + c = -\frac{k}{2}$. It follows that $b = k - (a + c) = \frac{3k}{2}$

Now, let's look at P(2)+ P(-2):

$$\underline{P(2) + P(-2) = 8a + 4b + 2c + k - 8a + 4b - 2c + k = 8b + 2k = 8\left(\frac{3}{2}k\right) + 2k = \mathbf{14k}}$$

Recitation #14 Function Solutions

1) Let $f(x) = x^2 - 4x + 7$, with a domain of $x \leq 2$. What is $f^{-1}(x)$? What is $f^{-1}(x)$'s domain and range?

Solution

Switch x and y and solve for y:

$$x = y^2 - 4y + 7$$
$$x - 3 = y^2 - 4y + 4$$
$$x - 3 = (y-2)^2$$
$$y - 2 = \pm\sqrt{x-3}$$
$$y = 2 \pm \sqrt{x-3}$$

Since the domain of the original function is $x \leq 2$, this correspond to the range of the inverse function. In order for this to be a function, we must choose either the + or the – sign, but not both. The latter choice restricts the range to $y \leq 2$, as desired. Thus, we have:

$$\underline{\boldsymbol{f^{-1}(x) = 2 - \sqrt{x-3}}}$$

2) Let $f(x) = 2^{3x-7}$ and $g(x) = 2x^2 + 5$. What are both $f(g(x))$ and $g(f(x))$? What is the minimum value of $f(g(x))$?

Solution

$$\underline{f\big(g(x)\big) = 2^{3(2x^2+5)-7} = \boldsymbol{2^{6x^2+8}}}$$
$$\underline{g\big(f(x)\big) = 2(2^{3x-7})^2 + 5 = \boldsymbol{2(4^{3x-7}) + 5}}$$

3) Let $f : A \to B$ and $g : B \to A$ be functions such that

(i) $\forall y \in B, (f \circ g)(y) = y$
(ii) $\exists\, x \in A : (g \circ f)(x) \neq x$

Prove that f is surjective (onto), but not injective (one-to-one).

<u>Solution</u>
f is surjective:
We need to prove that $\forall y \in B, \exists\, x \in A : f(x) = y$.
Let $y \in B$ be arbitrary. Consider the element $g(y)$. Let $x = g(y) \in A$.Then, $f(x) = f(g(y)) = y$ (by condition (i)). So, for all element $y \in B$, there exists $x \in A$ (in particular $x = g(y)$) such that $f(x) = y$.

f is not injective:
We need to prove that $\exists x_1, x_2 \in A : x_1 \neq x_2$ and $f(x_1) = f(x_2)$.
By condition (ii), $\exists\, x \in A : (g \circ f)(x) \neq x$. Let $x_1 = x$ and $x_2 = (g \circ f)(x)$. Note that $x_1 \neq x_2$.

Then,
$f(x_2) = f\Big(g\big(f(x)\big)\Big)$ (by definition of x_2)
$= (f \circ g)(f(x))$ (by definition of function composition)
$= f(x)$ (by condition (i))
$= f(x_1)$. (by definition of x_1).

Thus, there exist $x_1, x_2 \in A : x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

4) Let $f : A \to B$ and $g: B \to A$ denote two functions. Suppose $g \circ f(x) = x$ for all $x \in A$. Answer the following two questions:

a) Prove that the function f is an injection. (**Hint:** Prove that if $f(x) = f(y)$, then $x = y$.)

b) Prove that the function g is a surjection. (**Hint:** Let $y \in A$, we need to find $x \in B$ such that $g(x) = y$. Try $x = f(y)$.)

<u>Solution</u>

Let $f : A \to B$ and $g: B \to A$ denote two functions. Suppose $g \circ f(x) = x$ for all $x \in A$. Answer the following two questions:

(a) Prove that the function f is an injection.

Proof: We need to prove that if $f(x) = f(y)$ --- (1), then $x = y$ --- (2).

From (1), $g(f(x)) = g(f(y))$ because g is a function. Thus, $g \circ f(x) = g \circ f(y)$ --- (3) because $g(f(x)) = g \circ f(x)$ and $g(f(y)) = g \circ f(y)$. Since $g \circ f(x) = x$ for all $x \in A$ by assumption, so (3) implies $x = y$, which proves (2).

(b) Prove that the function g is a surjection.

Proof: Let $y \in A$, we need to find $x \in B$ such that $g(x) = y$.

Let $x = f(y)$. Note that $g(x) = g(f(y))$ by substitution

$= g \circ f(y)$ by the definition of $g \circ f$

$= y$, by the assumption that $g \circ f(x) = x$ for all $x \in A$

5) Let f(x) = 3x – 2, for all x < 3

= x + 4, for all x ≥ 3.

Prove or disprove: f(x) is a bijection over R → R.

This function is a bijection. To prove it, we must prove that it's both a surjection and an injection. Let's prove that it's a surjection. Consider an arbitrary value y. We must show that there exists an x such that f(x) = y. We split into two cases:

If y ≥ 7, then x = y – 4 is such that f(x) = y. Notice that if y ≥ 7, we must have f(x) = x + 4, since the maximum value of f(x) for x < 3 is y < 7. Plugging in, we get f(y – 4) = (y – 4) + 4 = y, as desired.

If y < 7, then x = $\frac{y+2}{3}$. We see that if y < 7, then f(x) = 3x – 2, since f(x) = x + 4 only for x ≥ 3. It follows that f($\frac{y+2}{3}$) $= 3\left(\frac{y+2}{3}\right) - 2 = y + 2 - 2 = y$, proving that the function is surjective.

To prove that the function is injective, we must show that if f(a) = f(b), then a ≠ b. Based on the domain restrictions, we know that exactly one of a and b are less than 3, then f(a) ≠ f(b) holds because 3x – 2 < 7 while x + 4 ≥ 7 for all x less than 3 and greater than or equal to 3, respectively. Thus, if the statement isn't true, either both must be less than 3 or both must be greater than 3. Let's try out both possibilities. If both are less than 3, we use the first expression for the function for both a and b:

3a – 2 = 3b – 2
3a = 3b
a = b, as desired

Now, let's consider the case where both are 3 or greater:

a + 4 = b + 4
a = b, as desired.

In all cases, we find that if f(a) = f(b), it must be the case that a = b. Thus, the given function is injective.