Recitation #13 Warm-Up Solutions
4/11/2014

1) The difference between a two-digit number and the number obtained by reversing its digits is 5 times the sum of the digits of either number. What is the sum of the two digit number and its reverse?

Solution

Let the number be ab and its reverse ba. These have values 10a + b and 10b + a, respectively. Using the given information, we obtain the equation

(10b + a) – (10a + b) = 5(a + b)
9b – 9a = 5a + 5b
4b = 14a
2b = 7a
Since the values are digits and their sum isn't zero, the only solution is a = 2, b = 7. The sum of the number and its reverse is 27 + 72 = **99.**

2) Five consecutive integers starting with *a* have an average of *b*. What is the average of 5 consecutive integers starting with *b*, in terms of *a*?

Solution

$$b = \frac{a + a + 1 + a + 2 + a + 3 + a + 4}{5} = a + 2$$

We can see that there's nothing unique about a here. Thus the desired average of the values requested in terms of b is b+2. Solving the original problem, we get b + 2 = (a + 2) + 2 = **a + 4.**

3) David drives from his home to the airport to catch a flight. He drives 35 miles in the first hour, but realizes that he will be 1 hour late if he continues at this speed. He increases his speed by 15 miles per hour for the rest of the way and arrives at the airport 30 minutes early. How many miles is the airport from his home?

Solution

Let d be the desired distance and let t equal the appropriate amount of time he has to get to the airport from when he begins. Using the first piece of information, we find that

d = 35(t+1)

Using the second piece of information, we get the equation

d – 35 = 50(t – 1.5)

Solve the system of equations:

d = 35t + 35
d = 35 + 50t – 75 = 50t – 40

35t + 35 = 50t – 40
15t = 75
t = 5

d = 35(5 + 1) = **<u>210 miles.</u>**

4) A fancy bed and breakfast inn has 5 rooms, each with a distinctive color-coded décor. One day 5 friends arrive to spend the night. There are no other guests that night. The friends can room in any combination they wish, but with no more than 2 friends per room. In how many ways can the innkeeper assign the guests to the rooms? (Note: The total number of assignments without the maximum number of friends in a room restriction is 5^5, because we are counting each person and each room as distinguishable. So, each unique room placement can be expressed as a ordered 5-tuple of rooms, where the i^{th} item in the tuple is the room the i^{th} person is staying in.)

<u>Solution</u>
Split our counting into the following cases:

a) Use 5 rooms
b) Use 4 rooms
c) Use 3 rooms

In the first case there are 5! = 120 arrangements, since each person has to pick a unique room.

In the second case, we have one room with two guests. We can select which room this is in 5 ways. We can select which two guests are in that room in $\binom{5}{2} = 10$ ways. The next three guests can select their room in 4, 3 and 2 ways respectively. Multiplying, we get 5 x 10 x 4 x 3 x 2 = 1200 ways we can place the guests into exactly four rooms.

In the last case, we choose two rooms to house 2 guests each in $\binom{5}{2} = 10$ ways. We then choose the pair for the first room in in $\binom{5}{2} = 10$ ways and the pair for the second room in $\binom{3}{2} = 3$ ways. Finally, this leaves the last individual 3 choices for their room. Multiplying, we get 10 x 10 x 3 x 3 = 900 ways we can place the guests into exactly 3 rooms.

Summing we get 120 + 1200 + 900 = **<u>2220</u>** total arrangements.

5) The product 8x(8888888…888), where the second factor has k digits, is an integer whose digits have a sum of 1000. What is k?

Using regular multiplication, we see our result will be of the form 7111…11104, where the number of digits is one more than k, so there are k – 2 1s. Thus, the sum of these digits is 7 + 4 + k – 2. Setting this to 1000, we find k = **<u>991.</u>**

Recitation #13 Relations Problems

1) If A is the set {1, 2, 3, 4} give an example of a relation on A that is:

a) reflexive and symmetric, but not transitive.
b) reflexive and transitive, but not symmetric.
c) symmetric and transitive, but not reflexive.

<u>Solution</u>

a) {(1,1), (2,2), (3,3), (4,4), (1,2), (2,1), (2,3), (3,2)}
This relation is reflexive because all the elements in A are paired with themselves in R.
The relation is symmetric because whenever (a,b) is in the set, so is (b,a).
The relation is not transitive because (1,2) and (2,3) are elements of R but (1,3) is not.

b) {(1,1), (2,2), (3,3), (4,4), (1,2)}
This relation is reflexive because all the elements in A are paired with themselves in R.
The relation is transitive because for all a, b, and c whenever (a,b), (b,c) appear, so does (a,c).
The relation is not symmetric because (1,2) is in the set and (2,1) is not.

c) {(1,1), (2,2), (1,2), (2,1)}
The relation is symmetric because whenever (a,b) is in the set, so is (b,a).
The relation is transitive because for all a, b, and c whenever (a,b), (b,c) appear, so does (a,c).
The relation is not reflexive because it does not contain (3,3) or (4,4).

2) Prove that the following relation R defined over $Z \times Z$ is transitive:

$$R = \{(x, y) \mid x + 3y = 4c, \text{ for some integer } c.\}$$

We must show that if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. Using the given information, we have

$$x + 3y = 4a, \text{ for some integer a}$$
$$y + 3z = 4b, \text{ for some integer b}$$

Now, multiply the second equation by -3 to yield $-3y - 9z = -12b$ and add this to the first equation:

$$(x + 3y) + (-3y - 9z) = 4a - 12b$$
$$x - 9z = 4a - 12b$$
$$x - 9z + 12z = 4a - 12b + 12z$$
$$x + 3z = 4(a - 3b + 3z)$$

Since x, z, a and b are all integers, it follows that x + 3z is a multiple of 4, as desired.

We can conclude that $(x, z) \in R$, as desired, showing that the relation is transitive.

3) Prove that a relation $R \subseteq A \times A$ is transitive if and only if $R^2 \subseteq R$.

<u>Solution</u>

First we will show that $R \subseteq A \times A$ is transitive $\Rightarrow R^2 \subseteq R$.

Let (x,y) be an arbitrary element of R^2. We must show that it is also an element of R.

If (x,y)$\in R^2$, by the definition of relation composition, we have that there exists an element z$\in$A such that (x,z)$\in R$ and (z,y)$\in R$. Since we are given that R is transitive, we can deduce that (x,y)$\in R$ as desired.

Now, let's show that if $R^2 \subseteq R \Rightarrow R$ is transitive.

We must show that for any arbitrary elements x, y and z, if (x,y)$\in R$ and (y,z)$\in R$, then (x,z)$\in R$.

If we have that (x,y)$\in R$ and (y,z)$\in R$, by definition of relation composition, it follows that (x,z)$\in R^2$. But, we are given that $R^2 \subseteq R$. So it necessarily follows that (x,z) $\in R$, which is what we needed to show.

4) Let a set A contain 10 elements. How many relations over $A \times A$ are reflexive and symmetric?

<u>Solution</u>

There are 100 elements in *A x A.* Of these, 10 are required to be in any reflexive relation. There are 90 elements of the form (x, y) which remain where x $\neq$ y and x and y are in A. These 90 can be paired in groups with two elements of the form (x, y) and (y, x). For each of the 45 groups, there are two choices – either putting both in the relation or not putting either in the relation, in terms of maintaining the symmetric property. We can make these choices in 2^{45} ways. Thus, there are <u>**2^{45}**</u> relations over *A x A* that are both reflexive and symmetric.

5) Let R = {(a, b) | a, b and b are positive integers that form the sides of a triangle} over the set Z^+. Determine, with proof, whether or not R is (a) reflexive, (b) symmetric, and (c) transitive.

Solution

R is reflexive because for all a, $(a,a) \in R$. To see this note that for any positive integer a, the three items a, a and a form a valid triangle. Specifically, they form an equilateral triangle.

R is not symmetric. Note that $(1, 2) \in R$ because we can form a triangle with side lengths 1, 2 and 2. (To see this is possible note that the two smallest sides add up to 3, which is bigger than the largest side.) But, $(2, 1) \notin R$ because no triangle can be formed with side lengths 2, 1 and 1, since 1 + 1 isn't greater than 2.

R isn't transitive. Note that $(4, 3) \in R$ and $(3, 2) \in R$, but $(4, 2) \notin R$. Specifically, we can make triangles with sides 4, 3 and 3 since $3 + 3 > 4$ and sides 3, 2 and 2, since $2 + 2 > 3$, but we can't make a triangle with sides 4, 2 and 2, because $2 + 2 \leq 4$.