Recitation #12 Warm-Up Solutions
4/4/2014

1) A particular 12-hour digital clock displays the hour and minute of a day. Unfortunately, whenever it is supposed to display a 1, it mistakenly displays a 9. For example, when it is 1:16 PM the clock incorrectly shows 9:96 PM. What fraction of the day will the clock show the correct time?

Solution
Of the 60 possible minute settings, the ones that show incorrectly are 01, 10 – 19, 21, 31, 41, and 51, for a total of 15 settings. Thus, there are a total of 45 correct settings for a probability of $\frac{3}{4}$ that the minute is correct. An easy way to count these is to use the inclusion exclusion principle. 10 settings have 1 in the units position and 6 settings have it in the tens position. Of all of these, one setting 11, is double counted since it has a 1 in both the tens and units place. Subtract this out to get 15. Of the 12 possible hours, 1, 10, 11 and 12 show incorrectly. Thus, 8 or $\frac{2}{3}$ of these show correctly. Multiplying since these two are independent, the fraction of settings that are correct is $\frac{2}{3} \times \frac{3}{4} = \mathbf{\frac{1}{2}}$.

2) The fifth and eighth terms of a geometric sequence of real numbers are 7! and 8! respectively. What is the first term?

Solution
Let the sequence be a_1, a_2, .. with common ratio r. We have $a_8 = 8! = 8(7!) = 9a_5$. Plugging in for both a_5 and a_8, we get $a_1r^7 = 9a_1r^4$. Simplifying, we find $r^3 = 8$, so r = 2. It follows that

$a_5 = 7! = 5040 = a_1 2^4$, dividing, we get $\mathbf{a_1 = 315}$.

3) Each face of a cube is given a single narrow stripe painted from the center of one edge to the center of its opposite edge. The choice of the edge pairing is made at random and independently for each face. What is the probability that there is a continuous stripe encircling the cube?

Solution
Pick the top face to fix, with its stripe going left to right. There is a $\frac{1}{8}$ chance that the two side faces and bottom face stripes would form a continuous stripe, since all three have to line up in one particular way out of two. The only other way a continuous stripe could form is by not using either the top or bottom face, going all the way around the other four faces, horizontal to the ground. The chance of this arrangement is $\frac{1}{16}$, since all four have to be lined up in one specific way out of two. Adding these two disjoint cases gives us a probability of $\mathbf{\frac{3}{16}}$.

4) In a certain year the price of gasoline rose by 20% during January, fell by 20% during February, rose by 25% during March and fell by x% during April. The price of gasoline at the end of April was the same as it had been at the beginning of January. What is x?

Solution

Let P be the original price. The we get the equation:

$P(1.2)(1 - .2)(1.25)(1 – x/100) = P$

$(1.2)(0.8)(1.25)(1 – x/100) = 1$

$1.2(1 – x/100) = 1$

$1 – x/100 = 5/6$

$1/6 = x/100$

$x = \mathbf{16\frac{2}{3}}\%$

5) Rachel and Robert run on a circular track. Rachel runs counterclockwise and completes a lap every 90 seconds, and Robert runs clockwise and completes a lap every 80 seconds. Both start from the start line at the same time. At some random time between 10 minutes and 11 minutes after they begin to run, a photographer standing inside the track takes a picture that shows one-fourth of the track, centered on the starting line. What is the probability that both Rachel and Robert are in the picture?

Solution

After 10 minutes have elapsed, Rachel will have run 6 2/3 laps, since she takes 1 ½ minutes to run a single lap. At the same time, Robert will have run 7 ½ laps, since he takes 1 1/3 minutes to run a single lap. Rachel will need to run 7/8 – 2/3 = 5/24 of a lap to get into view while Robert will have to run 7/8 – ½ = 3/8 of a lap to get into view. Rachel takes slightly less than 20 seconds while Robert takes 30 seconds to get into view. Thus, both are in the camera's view 30 seconds into the viewing window. Rachel exits the window after running 9/8 – 2/3 = 11/24 of a lap, which takes her 90 x 11/24 = 41 ¼ seconds. Thus, both are in the camera's view for 11 ¼ seconds out of 60, for a probability of **3/16.**

Recitation #12 Probability Problems

1) Disease A occurs in 0.02% of the population. If a person does NOT have the disease and takes a test for the disease, the test correctly indicates that they don't have the disease 99% of the time. If a person has the disease and takes the same test, the test correctly indicates that they do have the disease 95% of the time. Given that you've taken the test and have tested positive for disease A, what is the probability you actually have disease A? Given that you've taken the test and have tested negative for disease A, what is the chance that you have the disease anyway?

<u>Solution</u>

Draw the probability tree to get that the chance of having the disease and testing positive for it is (.0002)(.95) = .00019 and the chance of not having the disease and testing positive for it is (.9998)(.01) = .009998. The desired conditional probability is $\frac{.00019}{.00019+.009998} \sim .0186$. The probability of testing negative but having the disease is (.0002)(.05) = .00001. The probability of not having the disease and testing negative for it is (.9998)(.99) = .989802. If you've tested negative, the chance that you have the disease is $\frac{.00001}{.00001+.989802} \sim 1.01 \times 10^{-5}$.

2) Anderson gets 80% of the multiple choice questions in COT 3100 he answers. Given a set of 7 questions, what is the chance that he gets at least 5 of them? Write your answer using combinations and then use a calculator to get a decimal approximation for it.

<u>Solution</u>

Using the binomial distribution, let's sum over Anderson getting 5, 6 and 7 questions correct:

$$\binom{7}{5}(.8)^5(.2)^2 + \binom{7}{6}(.8)^6(.2)^1 + \binom{7}{7}(.8)^7(.2)^0 = .851968$$

3) If A and B are events and p(A) = 8/15, p(A ∩ B) = 1/3, p(A | B) = 4/7 calculate p(B), p(B|A) and p(B |$\bar{A}$), are A and B independent? Mutually exclusive?

<u>Solution</u>

$$p(A \mid B) = \frac{4}{7} = \frac{p(A \cap B)}{p(B)} = \frac{1/3}{p(B)}, \text{ thus } p(B) = \frac{1/3}{4/7} = \frac{7}{12}.$$

$$p(B \mid A) = \frac{p(A \cap B)}{p(A)} = \frac{1/3}{8/15} = \frac{5}{8}.$$

$$p(B \mid \bar{A}) = \frac{p(\bar{A} \cap B)}{p(\bar{A})} = \frac{p(B) - P(A \cap B)}{1 - p(A)} = \frac{7/12 - 1/3}{1 - 8/15} = \frac{1/4}{7/15} = \frac{15}{28}.$$

A and B aren't independent, since p(B | A) ≠ p(B) and p(A | B) ≠ p(A).
A and B aren't mutually exclusive since p(A ∩ B) ≠ 0.

4) A bag of popping corn contains 2/3 white kernels and 1/3 yellow kernels. Only ½ of the white kernels will pop, whereas 2/3 of the yellow ones will pop. A kernel is selected at random from the bag, and pops when placed in the popper. What is the probability that the kernel selected was white?

Solution

Draw a probability tree of the situation. Let Y = yellow, W = white, P = pop, N = did not pop.

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              *
 Y 1/3   /      \ W 2/3
        *         *
P 2/3  /  \       /  \ P 1/2 (the inner branches are for not popping)
    YP YN  WN WP
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From this tree, we have p(Y ∩ P) = $\frac{1}{3}\times\frac{2}{3}=\frac{2}{9}$, p(Y ∩ N) = $\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$,

p(W ∩ P) = $\frac{2}{3}\times\frac{1}{2}=\frac{1}{3}$, p(W ∩ N) = $\frac{2}{3}\times\frac{1}{2}=\frac{1}{3}$,

We need to determine p(W | P):

$$p(W\mid P)=\frac{p(W\cap P)}{p(P)}=\frac{1/3}{p(W\cap P)+p(Y\cap P)}=\frac{1/3}{1/3+2/9}=\frac{1/3}{5/9}=\frac{3}{5}.$$

5) Six distinct integers are picked from the set {1, 2, 3,…, 10}. What is the probability that among those selected, the second smallest is 3?

Solution

There are $\binom{10}{6}$ = 210 ways to pick 6 integers out of 10. Of these, we must count how many of these combinations of 6 have 3 as the second smallest value. In order for this to occur, we must choose 1 value from the set {1,2} and 4 values from the set {4, 5, 6, 7, 8, 9, 10}. This can be done in $\binom{2}{1}\times\binom{7}{4}=70$ ways. (We multiply because each choice from the first set can be paired up with any of the choices from the second set.) Thus, the desired probability is $\frac{70}{210}=\frac{1}{3}$.

6) Two machines, A and B, manufacture a particular component. Here is the chart showing each machine's effectiveness:

Machine	# Components	Probability Faulty
A	2500	0.04
B	1500	0.05

If a component is chosen randomly from all of them, what is the probability it is faulty? If a component chosen at random is found to be faulty, what is the probability that it was produced by machine A?

<u>Solution</u>
There are a total of 4000 components, of which

2500(.04) + 1500(.05) = 175 are faulty, so the probability of choosing a faulty component at random is 175/4000 = 7/160.

Now, we want the probability that, given a component is faulty, that it was produced by machine A. Thus, we simply need the probability that a component is faulty AND produced by machine A. There are 2500(.04) = 100 such components out of the total 4000. Thus, the desired probability is

(1/40) / (7/160) = 4/7.

It also follows that, given that a component is faulty, the probability it was produced by machine B is 1 – 4/7 = 3/7.

7) A bag contains 8 white balls and 12 black balls. Three balls are drawn from the bag, in sequence. Given that the second ball picked was white, what is the probability that all three balls were white?

<u>Solution</u>
Out of the eight possible orderings of pulling the balls, the sample space includes four of them:

BWB, BWW, WWB, WWW.

Here are the probabilities of each: p(BWB) = $\frac{12}{20} \times \frac{8}{19} \times \frac{11}{18}$, p(BWW) = $\frac{12}{20} \times \frac{8}{19} \times \frac{7}{18}$, p(WWB) = $\frac{8}{20} \times \frac{7}{19} \times \frac{12}{18}$, and p(WWW) = $\frac{8}{20} \times \frac{7}{19} \times \frac{6}{18}$. Summing these we get a probability of $\frac{38}{95}$. To get the conditional probability we calculate $p(WWW|s = W) = \frac{p(WWW)}{p(s=W)} = \frac{14/285}{38/95} = \frac{14}{114}$.