

**Recitation #11 Warm-Up Solutions**  
**3/28/2014**

1) For each positive integer  $n$ , the mean of the first  $n$  terms of a sequence is  $n$ . What is the 2008<sup>th</sup> term of the sequence?

Solution

This means that for each positive integer  $n$ , the sum of the first  $n$  terms must be  $n^2$ . Thus, the sum of the first 2008 terms is  $2008^2$  and the sum of the first 2007 terms is  $2007^2$ . The difference of these two sums is the value of the 2008<sup>th</sup> term:  $2008^2 - 2007^2 = (2008 - 2007)(2008 + 2007) = \mathbf{4015}$ .

2) Bricklayer Brenda takes 9 hours to build a chimney alone, and bricklayer Brandon takes 10 hours to build it alone. When they work together, they talk a lot and their output decreases by 10 bricks per hour, total. Working together, they build a chimney in 5 hours. How many bricks are in the chimney?

Solution

Let  $C$  = the number of bricks in a Chimney. Brenda lays  $C/9$  bricks an hour and Brandon lays  $C/10$  bricks an hour. Together they lay  $C/9 + C/10 - 10$  bricks an hour. If it takes them 5 hours to finish the chimney, we have the following:

$$\begin{aligned} 5\left(\frac{C}{9} + \frac{C}{10} - 10\right) &= C \\ \frac{95C}{90} - 50 &= C \\ \frac{C}{18} &= 50 \\ \mathbf{C} &= \mathbf{900} \end{aligned}$$

3) A rectangular floor measures  $a$  feet by  $b$  feet, where  $a$  and  $b$  are positive integers with  $b > a$ . An artist paints a rectangle on the floor with the sides of the rectangle parallel to the sides of the floor. The unpainted part of the floor forms a border of width 1 foot around the painted rectangle and occupies half of the area of the entire floor. List all the possible ordered pairs  $(a, b)$ .

Solution

The area of the painted portion of the floor is  $(a - 2)(b - 2)$ . Using the given information:

$$\begin{aligned} 2(a - 2)(b - 2) &= ab \\ 2ab - 4a - 4b + 8 &= ab \\ ab - 4a - 4b + 8 &= 0 \\ a(b - 4) &= 4b - 8 \\ a &= \frac{4(b-1)}{b-4} = 4 + \frac{8}{b-4}, \text{ via division} \end{aligned}$$

We need to set  $b$  such that  $b - 4$  is a positive divisor of 8 AND such that  $a < b$ . Of the four divisors of 8, two of them,  $b - 4 = 4$  and  $b - 4 = 8$  lead to valid solutions **(6, 8)** and **(5, 12)**.

4) Points A and B are on a circle with radius 5 and  $AB = 6$ . Point C is the midpoint of the minor arc AB. What is the length of the line segment AC?

Solution

Since  $\overline{AB}$  has length 6, the radius drawn from O, the center of the circle to C, bisects AB. Let this intersection point be D. Then  $AD = DB = 3$  and both ADO and BDO are 3-4-5 right triangles. Let  $\theta$  be angle AOC. It follows that  $\sin\theta = \frac{AD}{AO} = \frac{3}{5}$  and  $\cos\theta = \frac{DO}{AO} = \frac{4}{5}$ . To calculate the length of AC, we can just use the law of cosines:

$$AC^2 = AO^2 + CO^2 - 2(AO)(CO)(\cos\theta)$$

$$AC^2 = 5^2 + 5^2 - 2(5)(5)\left(\frac{4}{5}\right)$$

$$AC^2 = 10$$

$$AC = \sqrt{10}$$

Note: A easier solution is simply noting that OC is a radius, so DO is length 1 and then ADC is a right triangle with legs 1 and 3. It follows that AC, its hypotenuse, is  $\sqrt{3^2 + 1^2} = \sqrt{10}$ .

5) A circle has a radius of  $\log_{10}(a^2)$  and a circumference of  $\log_{10}(b^4)$ . What is  $\log_a b$ ?

Solution

Let  $r$  be the radius of the circle and  $c$  be its circumference. We have the following:

$$r = 2\log_{10}a$$

$$c = 2\pi r = 4\log_{10}b$$

We can equate expressions for  $r$  getting

$$2\log_{10}a = 2\left(\frac{\log_{10}b}{\pi}\right)$$

$$\pi = \left(\frac{\log_{10}b}{\log_{10}a}\right)$$

$$\log_a b = \pi$$

## Recitation #11 Counting Solutions

1) The dean of a certain college has a pool of 10 chemists, 7 psychologists, and 3 statisticians from which to form a committee.

a) The chemist, Professor C, and the psychologist, Professor P, do not get along. How many 5-member committees can be formed that do not have both Professor C and Professor P on them?

### Solution

We can use the subtraction rule for this one. There are  $10 + 7 + 3 = 20$  professors. Note that the total number of combinations to pick 5 members (disregarding the rule) is  $\binom{20}{5}$ . Now, we need to determine how many possible combinations there are for making a 5-member committee that INCLUDES both of these professors. This is similar to **4a**, and is  $\binom{18}{3}$ . So now we just use our subtraction rule to get our

final answer:  $\binom{20}{5} - \binom{18}{3}$

b) How many 5-member committees can be formed so that the number of chemists is greater than or equal to the number of psychologists and the number of psychologists is greater than or equal to the number of statisticians?

### Solution

We will use our addition rule to add up the different possible cases. Note that for the different cases the numbers of professors must add up to 5.

Case 1) All chemists. This would obey our rule and would simply be  $\binom{10}{5}$

Case 2) 4 chemists; so there must be 1 psychologists. So we have to choose 4 chemists from the 10 and 1 psychologist from the 7. This case will be the product of the two combinations, so it's  $\binom{10}{4} \binom{7}{1}$

Case 3) 3 chemists and 2 psychologists:  $\binom{10}{3} \binom{7}{2}$

Case 4) 3 chemists, 1 psychologist and 1 statistician:  $\binom{10}{3} \binom{7}{1} \binom{3}{1}$

Case 5) 2 chemists, 2 psychologist and 1 statistician:  $\binom{10}{2} \binom{7}{2} \binom{3}{1}$

So the final answer is the sum of these:  $\binom{10}{5} + \binom{10}{4} \binom{7}{1} + \binom{10}{3} \binom{7}{2} + \binom{10}{3} \binom{7}{1} \binom{3}{1} + \binom{10}{2} \binom{7}{2} \binom{3}{1}$

2) Johnny lives in New York City on a grid system. His apartment is at the corner of 25th Street and 8th Avenue while his work place is located at 42nd Street and 2nd Avenue. Streets and Avenues are present at every block.

a) If Johnny always walks either North (street numbers are increasing) or East (avenue numbers are decreasing), in how many ways can he walk to work?

Solution

The number of steps on the North direction Johnny must take = 17

The number of steps on the East direction Johnny must take = 6

There are a total of 23 steps Johnny will have to take to arrive to his destination. From the 23, he can choose to go east any time as long as he does 6 times. Thus, the total number ways he can go to work is

given by  $\binom{23}{6}$ .

b) If Johnny wants to get coffee at the intersection of 40th Street and 6th Avenue on his way to work, how many ways can he go?

Solution

From home to the coffee place:

The number of steps on the North direction Johnny must take = 15

The number of steps on the East direction Johnny must take = 2

The total number ways he can go to the coffee place is  $\binom{17}{2}$ .

From the coffee place to work:

The number of steps on the North direction Johnny must take = 2

The number of steps on the East direction Johnny must take = 4

The total number ways he can go to the coffee place is  $\binom{6}{4}$ .

Thus, the total number of ways he can go to work and stopping for coffee on his way is  $\binom{17}{2} \binom{6}{4}$ .

c) If Johnny wants to avoid the intersection of 40th Street and 6th Avenue on his way to work, how many ways can he go?

Solution

Since there are  $\binom{17}{2} \binom{6}{4}$  total paths that allow Johnny to stop by the coffee place, the number of paths

that avoid the coffee place must be  $\binom{23}{6} - \binom{17}{2} \binom{6}{4}$ .

d) If Johnny wants to avoid the intersection mentioned in part (c) AND also avoid the intersection of 33rd Street and 7th Avenue on his way to work, how many ways can he go?

Solution

We can proceed as in part c). Let us calculate the number of paths that cross the intersection of 33rd Street and 7th Avenue.

From home to the intersection of 33rd Street and 7th Avenue:

The number of steps on the North direction Johnny must take = 8

The number of steps on the East direction Johnny must take = 1

The total number ways he can go to the intersection of 33rd Street and 7th Avenue is  $\binom{9}{1}$ .

From the intersection of 33rd Street and 7th Avenue to work:

The number of steps on the North direction Johnny must take = 9

The number of steps on the East direction Johnny must take = 5

The total number ways he can go to work from the intersection of 33rd Street and 7th Avenue is  $\binom{14}{5}$ .

The total number ways he can go to work crossing the intersection of 33rd Street and 7th Avenue is  $\binom{9}{1}\binom{14}{5}$ .

But, there are a few paths we have double counted, because they cross through both of these intersections. Utilizing the inclusion-exclusion principle, we must add them back in. To count these paths, multiply the number of ways we can get to the 33rd street intersection, by the number of ways we can get to the 40th street intersection by the number of ways we can get to work:

$$\binom{9}{1}\binom{8}{1}\binom{6}{4} = 1080$$

Thus, the total number of ways he can avoid the intersection where the coffee place is and the intersection of 33rd Street and 7th Avenue is

$$\binom{23}{6} - \binom{17}{2}\binom{6}{4} - \binom{9}{1}\binom{14}{5} + \binom{9}{1}\binom{8}{1}\binom{6}{4}$$

3) How many non-negative solutions are there to the equation  $a + b + c + d + e = 27$  with  $a > 5$ ?

Solution

This is a combination with repetition problem. We only need to deal with the restriction on  $a$ . Just set  $a = a' + 6$ . Thus,  $a' \geq 0$ . Then, we simply want to know the number of non-negative solutions to  $a' + b + c + d + e = 21$ . Using the combinations with repetition formula with 5 distinct items chosen out of 21 total we get a solution of  $\binom{21 + 5 - 1}{5 - 1} = \binom{25}{4}$ .

4) A farmer wants to plant 3 birch trees, 4 oak trees and 5 palm trees in a row such that no two palm trees are adjacent. How many ways can he do it?

Solution

Create the 7 non-palm trees as our separators. Let these be T1 through T7:

\_\_\_ T1 \_\_\_ T2 \_\_\_ T3 \_\_\_ T4 \_\_\_ T5 \_\_\_ T6 \_\_\_ T7 \_\_\_

We can choose 5 of the 8 blank spots to place the palm trees. This can be done in  $\binom{8}{5}$  ways. The remaining 7 trees can be placed in  $\binom{7}{3}$ , by choosing 3 of the 7 given slots for the birch trees. The remaining 4 slots are fixed to be oak trees. Thus, the total number of possible orderings is  $\binom{8}{5} \binom{7}{3} = 1960$ .

5) How many integers between 1 and 1,000,000 have a sum of digits equal to 9? For example, 333 and 10,125 are valid, but 9,999 and 12,111,111 are invalid.

Solution

Let the digits of the number be  $a, b, c, d, e$  and  $f$ . Thus, we want the number of solutions to  $a + b + c + d + e + f = 9$ , where  $a, b, c, d, e,$  and  $f$  are non-negative digits. This is identical to the combinations with repetition problem. Thus, the number of solutions is  $\binom{9 + 6 - 1}{6 - 1} = \binom{14}{5}$ .