Recitation #10 Warm-Up Solutions
3/21/2014

1) A teacher gave a test to a class in which 10% if the students are juniors and 90% are seniors. The average score on the test was 84. The juniors all received the same score, and the average score of the seniors was 83. What score did each junior receive on the test?

Solution

Let S = the average score of the seniors and J = the average score of the juniors. We then have:

$.9S + .1J = 84$

Substituting, we get

$.9(83) + .1J = 84$
$.1J = 9.3$
$J = 93$

Thus, each junior got a 93.

2) A geometric series $a + ar + ar^2 + \ldots$ has a sum of 7, and the terms involving odd powers of r have a sum of 3. What are the values of a and r?

Solution

Using the sum of an infinite geometric series, we get $\frac{a}{1-r} = 7$. Now, consider the infinite geometric series of even terms, which must add to 4. Its common ratio is just r^2. It follows that summing this series, we find: $\frac{a}{1-r^2} = 4$. We can rewrite this second equation as follows:

$$\frac{a}{(1-r)} \times \frac{1}{(1+r)} = 4$$

We can directly substitute from the first equation for the term on the left hand side in the product as follows:

$7 \times \frac{1}{(1+r)} = 4$, cross-multiplying, we get $1 + r = \frac{7}{4}$, so $\boldsymbol{r = \frac{3}{4}}$**.** It follows that $\boldsymbol{a = \frac{7}{4}}$ via substitution into the first equation.

3) If a is a nonzero integer and b is a positive number such that $ab^2 = \log_{10}b$, what is the median of the set {0, 1, a, b, 1/b}?

Solution

First notice that we can't have b = 1, since in that situation a = 0 and we are given that $a \neq 0$. Thus, it follows that exactly 1 of b and 1/b is greater than 1, since b is positive. Notice that $a = \frac{log_{10}b}{b^2}$. For all b > 1, the value on the right is greater than 0 (since both the numerator and denominator are positive) and less than 1 (since $b^2 > \log_{10}b$ on this interval.) Thus, there are no solutions with b > 1. All solutions have 0 < b < 1. In this case, the value on the right MUST be negative, since if 0 < b < 1, $\log_{10}b < 0$ and $b^2 > 0$. Thus, if we want to order the terms, a is the smallest, 0 is the second smallest, **b is the median**, 1 is the fourth smallest term, and 1/b is the largest term.

4) A traffic light repeatedly runs through the following cycle: 30 seconds green, 3 seconds yellow, 30 seconds red. Leah picks a random 3 second interval to watch the light. What is the probability that the color changes while she is watching?

Solution

Consider a single cycle, each time is equally likely to be a starting time for Leah's interval. Within the cycle, if she starts in the range t = 27 to t = 33, she'll either witness the change to yellow or the change to red. If she starts watching in the range t = 60 to t = 63, she'll see the light turn back green. Thus, out of 63 seconds, there are 9 seconds she could start watching which will involve a change in light color. The desired probability is **1/7.**

5) Let a, b and c be digits with $a \neq 0$. The three-digit number abc lines one third of the way from the square of a positive integer to the square of the next larger integer. The integer acb lies two thirds of the way between the same two squares. What is a + b + c?

Solution

Clearly, b < c. The value of the smaller value mentioned is 100a + 10b + c and the value of the larger value mentioned is 100a + 10c + b. The difference of these two values is divisible by 9 (it's 9c – 9b), so we know that our difference in squares is divisible by 9 x 3 = 27, since the difference between these values is a third of the difference between successive squares. Let n^2 be the smaller of the two squares in question. We then have

$(n + 1)^2 - n^2 = 2n + 1 \equiv 0 \pmod{27}$.

All vales of n that satisfy this relationship are n = 13 + 27a, $a \in \mathbb{Z}$. (We easily arrive at 13 as a solution, and since 2 and 27 are relatively prime, all other solutions must be created by adding multiples of 27 to our base solution.) Only one of these values n = 13 corresponds to a three digit number. Thus, the difference between our successive squares is 27 exactly (169 to 196), and the digits are a = 1, b = 7, c = 8, respectively. The sum of these digits is 16.

Recitation #10 Counting Solutions

1) Using the multiplication principle, determine the number of passwords of length 8 we can make where each character must be an uppercase letter. Imagine relaxing the length restriction so that a password could be 8 or fewer uppercase characters (but can't be the empty string). Write a summation equal to the valid number of passwords in this situation. Using the formula for a finite geometric sequence, simplify the summation you derived. Using the work in your example, determine the number of passwords of length N or fewer characters (with at least one character) where each character is chosen from one of c choices.

Solution

There are 26 uppercase letters. For each of 8 slots we can choose any of the 26 letters. Since each choice in one slot can be paired with every setting of the other slots, we want to multiply 26 by the number of settings in the rest of the slots. Iterating the multiplication principle, we get 26^8 possible passwords. For the second part, we simply want to sum over passwords of lengths 1 through 8, inclusive, arriving at the sum $\sum_{i=1}^{8} 26^i$. Using the geometric sum formula, we find this sum to equal $\frac{26(26^8-1)}{(26-1)} = \frac{26(26^8-1)}{25}$. To answer the general question, plug in the appropriate variables in the appropriate places to get $\frac{c(c^N-1)}{c-1}$.

2) Using the multiplication principle, determine the number of orderings of the numbers 1, 2, 3, and 4. Using similar thinking, generalize the result to determine the number of orderings of the numbers 1, 2, 3, …, n.

Solution

We have 4 choices for the first number, then 3 choices for the next (since one of them is invalid), followed by 2, then 1. Thus we have 4 x 3 x 2 x 1 = 24 or 4! orderings of the 4 numbers. In general, we'll have n! ways to order n distinct items.

3) Using the multiplication principle, determine how many unique lists of length 3 you can make from the numbers 1, 2, 3, 4, 5, where you are allowed to use each number at most one time. Using similar thinking, generalize the result to determine the number of unique lists of length k you can make from the numbers 1, 2, 3, …, n. You may assume that $1 \le k \le n$. Express your answer in pi notation. See if you can come up with an alternate expression equal to the same answer using two factorials in a fraction, one in the numerator and one in the denominator.

Solution

There are 5 choices for the first item, 4 choices for the second and 3 choices for the third. Thus, there are 5 x 4 x 3 = 60 unique lists of length 3 that can be formed from the numbers 1, 2, 3, 4, 5. If we generalize our problem, we get n x (n – 1) x (n – 2) x … (n – k + 1) possible unique lists. In formal product notation this is $\prod_{i=0}^{k-1}(n-i)$. If we take this expression and multiply it by $\frac{(n-k)!}{(n-k)!}$, we get the following $\prod_{i=0}^{k-1}(n-i) = \frac{(\prod_{i=0}^{k-1}(n-i))\prod_{i=1}^{n-k} i}{(n-k)!} = \frac{\prod_{i=1}^{n} i}{(n-k)!} = \frac{n!}{(n-k)!}$.

4) If we look at all of the lists in question three, we see that some lists, such as 1, 4, 3 and 3, 1, 4, both contain the exact same three elements. We can group the lists in question three into groups such that each group contains lists of the exact same three elements. How many lists are in each group? How many different groups are there? Each of these groups represents one combination of 3 items out of 5. Using our generalized answer to #3 and our generalizing our answer to the question, "How many lists are in each group?", we can determine a general answer to the question, "How many combinations of k items are there out of n?", in terms of both n and k. Give your answer to this question. You may use factorials, the pi symbol, or both.

Solution

Since our original system lists each possible ordering, there are 3! orderings of each group. Thus, using the division principle, there must be (5 x 4 x 3)/3! = 10 groups, or combinations of 3 items out of 5. The number of lists in each group is simply k!, since there are k! ways to order k distinct objects in a list, and each of these orderings will exist in the permutations listed. We can use the division principle again and find that the total number of combinations of k items out of n is $\frac{n!}{(n-k)!}\Big/k! = \frac{n!}{(n-k)!k!}$. This expression is so common in counting it has a specific name and symbol. It's a combination and we express it as follows: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

5) An ascending number is a number where each subsequent digit is strictly greater than the previous digit. For example, 237, 1389 and 45 are all ascending numbers. How many ascending numbers of length 5 are there?

Solution

Consider any combination of 5 of the 9 digits 1, 2, 3, 4, 5, 6, 7, 8, 9. Each combination can be used to create precisely one ascending number of length 5. Similarly, each ascending number of length 5 maps to a unique combination of 5 digits out of 9. Since we've established a one-to-one mapping the sizes of the sets are the same. There are $\binom{9}{5}$ such combinations by definition, which is also the number of ascending passwords of length 5.