

COT 3100 Homework #10 Solutions (by Arup Guha)

Relations and Functions

1)(3 pts) Let A, B be sets with $|B| = 5$. If there are 2^{30} relations from A to B , what is $|A|$?

Solution

The total number of relations from a set A to a set B is $2^{|A||B|}$ (**1 pt**), since each item in A can be mapped to any of the $2^{|B|}$ subsets of B . These choices are independent for each of the $|A|$ items in A , so we need to multiply $2^{|B|}$ by itself $|A|$ times, yielding $2^{|A||B|}$. Since $|B| = 5$, we must have that $|A| = 6$ (**2 pts**).

2)(6 pts – 2 pts, 4 pts)

- a) Give an example of a relation that is irreflexive and transitive, but not symmetric.
- b) Let R be a non-empty relation on a set A . Prove that if R satisfies any of the two following properties – irreflexive, symmetric, transitive – then it can not satisfy the third.

Solution

a) Let A be the set $\{0, 1\}$ and let R be a relation on A . Then the relation $R = \{(0,1)\}$ is irreflexive, transitive, and not symmetric.

The relation is irreflexive because $\forall x(x \in A \Rightarrow \neg xRx)$, which is true because $0R0$ and $1R1$ are both false. The relation is transitive because $\forall x \forall y \forall z((x \in A \wedge y \in A \wedge z \in A \wedge xRy \wedge yRz \Rightarrow xRz)$, which is true because the antecedent is always false. Finally, the relation is not symmetric because $\exists x \exists y(x \in A \wedge y \in A \wedge xRy \wedge \neg yRx)$, which is true because $0R1$ and not $1R0$. (**2 pts – 1 for the relation, 1 for showing it satisfies the requirements.**)

(Alternate) A second example is the less-than operator on the real numbers. The relation is irreflexive because no number is less than itself. The relation is transitive because if a number is less than something that is also less than something else, then the first number is also less than the third. Finally, the relation is not symmetric because, for example, $2 < 4$ and not $4 < 2$.

b) Assume that some relation R is irreflexive, symmetric, and transitive. Because R is nonempty and irreflexive, we know there is some x and y with $x \neq y$ that are related by R , so we can say xRy . (**1 pt**) Because R is symmetric, we can add that yRx . . (**1 pt**) And since R is transitive, from these we can see that xRx . . (**1 pt**) But because R is irreflexive, it is false that xRx . . (**1 pt**) So our assumption that R could be irreflexive, symmetric, and transitive is false. Therefore all relations must fail to satisfy at least one of the three properties.

3)(5 pts) Let $|A| = 12$. Determine the number of binary relations on $A \times A$ that satisfy the following properties:

- a) reflexive
- b) neither reflexive nor irreflexive
- c) symmetric
- d) reflexive and symmetric
- e) irreflexive and anti-symmetric

Solution

a) This is the same as counting the ways to pick any number of elements from a 12 by 102 matrix if we always need to pick the 12 elements along the diagonal. There are $12 * 12 - 12 = 132$ elements in $A \times A$ that can be either in the relation or not in it. There are 2^{132} different ways to define a relation using these elements. **(1 pt)**

b) For this relation, there are still 132 elements not on the diagonal and 2^{132} ways to arrange them. As for the 12 elements that are on the diagonal, there are $2^{12} - 1 - 1$ ways to arrange them without making the relation reflexive or irreflexive. So the total number of possible relations is $2^{132} * (2^{12} - 2) = 2^{144} - 2^{133}$. **(1 pt)**

c) For a symmetric relation, half of the 132 elements that are not on the diagonal are determined by the corresponding element on the other side of the diagonal. So there are 2^{66} ways to pick the elements not on the diagonal. The elements on the diagonal can have any value, so there are 2^{12} ways to pick them. There are $2^{66} * 2^{12} = 2^{78}$ possible relations. **(1 pt)**

d) By arguments similar to the above, there are 2^{66} ways to pick the elements outside of the diagonal, and the elements on the diagonal are always in the relation because the relation is reflexive. Therefore there are 2^{66} total relations possible. **(1 pt)**

e) There is only one way to pick the elements on the diagonal of $A \times A$ so that the relation is irreflexive. Since the relation is antisymmetric, pairs of elements outside of the diagonal can have one of three states: the element above the diagonal is in the relation, the element below the diagonal is in the relation, or neither is. So there are 3^{66} possible relations. **(1 pt)**

4)(8 pts – 1 pt, 2 pts, 5 pts) With proof, determine if the following relations are equivalence relations, partial ordering relations, or neither.

- a) $\{ (a, b) \mid a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, a > 2b \text{ or } b > 2a \}$
- b) $\{ (a, b) \mid a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, a \equiv 0 \pmod{b} \text{ or } b \equiv 0 \pmod{a} \}$
- c) $\{ (a, b) \mid a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, \text{ and } \exists c \in \mathbb{Z}^+ \text{ such that } b = ac \}$

Solution

a) This relation is neither an equivalence relation nor a partial order because it is not reflexive. For example, 1 is an element of the positive integers, but $(1,1)$ is not an element of the relation because it is not true that $1 > (2)(1)$. **(1 pt)**

b) This relation is reflexive, since for all positive integers a , $a \equiv 0 \pmod{a}$. It's also symmetric by definition because if (a, b) is in the relation, either $a \equiv 0 \pmod{b}$ or $b \equiv 0 \pmod{a}$. Whichever one of the two is true, by switching b and a , the other statement will become true. But, the statement isn't transitive. Consider the following counter-example: $(4, 2) \in R$, $(2, 6) \in R$, but $(4, 6) \notin R$. In a nutshell, 4 is divisible by 2 and 6 is divisible by 2, but this doesn't necessarily mean that 4 is divisible by 6 or that 6 is divisible by 4. **(2 pts – 1 pt answer, 1 pt justification)**

c) This relation is very, very similar to b in that it involves divisibility, but it's different because it forces the second item in the ordered pair to be the larger one. This should hint at the idea that this relation is a partial ordering relation. Let's prove it:

For all a , $a \mid a$, thus, the relation is reflexive. **(1 pt)**

We must now prove that it's anti-symmetric. Namely, we must show that if $(a,b) \in R$ and $(b, a) \in R$, then $a = b$.

The first statement implies that there is some positive integer c such that $b = ac$. The second statement implies that there is some positive integer d such that $a = db$. **(1 pt)** Substitute the expression for a in the first equation into the second to get:

$$b = (db)c, \text{ since } b \text{ isn't } 0, \text{ divide both sides by } b \text{ to get } 1 = cd$$

The only solution to this equation in the positive integers is $c = 1$, $d = 1$. It follows that $a = b$, as desired. **(1 pt)**

Now, to prove transitivity we must show that if $(a,b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

The first statement implies that there is a positive integer d such that $b = da$

The second statement implies that there is a positive integer e such that $c = eb$ **(1 pt)**

Substitute the expression for b in the first equation into the second one:

$$c = e(da)$$

$$c = (ed)a \text{ **(1 pt)**}$$

Since e and d are positive integers, so is ed . It follows that $(a, c) \in R$, by definition of R .

5)(6 pts) Let $g: A \rightarrow A$ be a bijection. For $n \geq 2$, define $g^n = g \circ g \circ \dots \circ g$, where g is composed with itself n times. Prove that for $n \geq 1$, that $(g^n)^{-1} = (g^{-1})^n$, by using induction on n .

Solution

B.C.: $n = 1$.

$$\Rightarrow (g^1)^{-1} = (g^{-1})^1 \Rightarrow (g)^{-1} = (g^{-1}) \Rightarrow g^{-1} = g^{-1}. \text{ (1 pt)}$$

I.H.: Assume for $n = k \geq 1$, $(g^k)^{-1} = (g^{-1})^k$. **(1 pt)**

I.S.: We need to prove for $n = k + 1$, $(g^{k+1})^{-1} = (g^{-1})^{k+1}$. **(1 pt)**

Note the LHS $= (g \circ g^k)^{-1}$, by the given definition

$$= (g^k)^{-1} \circ (g)^{-1}, \text{ by theorem about function inverse comp. (1 pt each step)}$$

$$= (g^{-1})^k \circ g^{-1}, \text{ by the I.H.}$$

$$= (g^{-1})^{k+1} = \text{RHS.}$$

6)(8 pts) Prove that following function is a bijection from the open interval $(0,5)$ to the positive real numbers:

$$f(x) = \frac{5-x}{5x}$$

Solution

To show that this function is a bijection, we need to show it's injective and surjective.

Proof of injection:

Note that $f(a) = \frac{5-a}{5a}$, and $f(b) = \frac{5-b}{5b}$. We now look at $f(a) = f(b)$ to determine if a and b are the same. **(1 pt)** We see:

$$f(a) = f(b) \Rightarrow \frac{5-a}{5a} = \frac{5-b}{5b} \Rightarrow 25b - 5ab = 25a - 5ab \Rightarrow 25b = 25a \Rightarrow a = b. \text{ Since } a \text{ and } b \text{ are}$$

the same, the function is injective. (Note: We can multiply through by a and b in the first step because we know that neither of them is 0.) **(3 pts algebra)**

Proof of surjection:

Let there exist an element $r \in \mathfrak{R}$. We need to prove that there exists $a \in (0,5)$ such that

$$f(a) = r. \text{ (1 pt)}$$

$$\text{Note } f(a) = \frac{5-a}{5a} = r \Rightarrow 5ar = 5-a \Rightarrow 5ar + a = 5 \Rightarrow a(5r+1) = 5 \Rightarrow a = \frac{5}{5r+1}.$$

Since $r \in \mathfrak{R}$, we know that the denominator of this is greater than 1, because the smallest value that $5r$ can be is still greater than zero. So when r is at its minimum value, then a will be less than 5. As the value of r becomes larger, the value of the denominator will grow. We must show though, that the value of a will never be greater than or equal to 5. Let x be the largest real number less than 5. If we plug in x for a we see:

$$x = \frac{5}{5r+1} \Rightarrow 5rx + x = 5 \Rightarrow x(5r+1) = 5 \Rightarrow 5r+1 = \frac{5}{x} > 1, \text{ because of our definition of } x.$$

So we see $5r + 1 > 1 \Rightarrow 5r > 0 \Rightarrow r > 0$, which agrees with the definition of r . Thus no matter what positive real value is picked for r , there is a value $a \in (0,5)$ that maps to it. **(3 pts)**

And by showing that the function is both injective and surjective, we've proven that it's bijective.

7)(5 pts) Determine the inverse of the function defined in question #6.

Solution

First swap the x and y : $x = \frac{5-y}{5y}$ **(1 pt)**

Then we solve for y : $x = \frac{5-y}{5y} \Rightarrow 5yx = 5 - y \Rightarrow 5yx + y = 5 \Rightarrow y(5x + 1) = 5 \Rightarrow y = \frac{5}{5x + 1}$, our inverse function. **(4 pts algebra)**

8) (5 pts) Let $f : A \rightarrow B$ and $g : B \rightarrow C$ denote two functions. Prove or disprove that if $g \circ f : A \rightarrow C$ is injective, then f is injective.

Solution

We will prove this by contradiction. Assume that f isn't injective. **(1 pt)** This implies that there exist some $a, b \in A$, where $a \neq b$, and $f(a) = f(b)$. **(1 pt)** $g \circ f$ being injective implies that $(g \circ f)(a) = (g \circ f)(b)$, iff $a = b$. **(1 pt)** But by our assumption we see that there are two elements a and b that are not equal, but $f(a) = f(b)$ which implies $g(f(a)) = g(f(b))$, giving us our contradiction. **(2 pts)**

9) (4 pts) Let $f : A \rightarrow B$ and $g : B \rightarrow C$ denote two functions. Prove or disprove that if $g \circ f : A \rightarrow C$ is injective, then g is injective.

Solution

We will disprove by counterexample.

Define f as $f : \{a\} \rightarrow \{a, b\}$, $f(a) = a$.

Define g as $g : \{a, b\} \rightarrow \{a\}$, $g(a) = a$, $g(b) = a$. **(2 pts)**

Then, we have $g \circ f = \{(a, a)\}$. **(1 pt)**

We see that $g \circ f$ is injective, but g isn't. **(1 pt)**