COT 3100 Homework #8 Solutions (by Maxwell Miller)
Counting

1) **(4 pts)** Consider an ant that is walking on a Cartesian grid, starting at (0,0) and ending at (10, 15). The ant always chooses to walk exactly one unit either up or to the right (towards his destination) whenever he arrives at a Lattice point. (A Lattice point is a point with integer coordinates.) Thus, from (0,0) he either walks to (1, 0) or (0, 1). How many different paths can he take on his walk?

Solution
Let the character 'R' represent moving to the right, and 'U' as moving up on the grid. Since, in order to reach the goal, you need to move right 10 times, and up 15 times, the problem reduces to finding the number of rearrangements of a string containing 10 'R's and 15 'U's. The number of such arrangements is $\frac{25!}{10!15!} = 3{,}268{,}760$. **(4 pts – 2 pts for values, 2 pts for combo/perm with rep)**

25! Represents the number of total arrangements, the denominator removes all of the duplicate strings caused by repeated 'R's and 'U's, as every unique arrangement will be duplicated for each different possible arrangement of 'R's and 'U's (which are 10! and 15! respectively).

2) (13 pts) This question considered permutations of "ENGINEERING"

a) How many permutations are there total?

Solution
Since we have 11 letters, there are 11! arrangements of them, not considering duplicates. There are 3 duplicates of 'E', 2 duplicates of 'I', 2 duplicates of 'G', and 3 duplicates of 'N', so we divide by the number of arrangements of those to remove the over counting they caused. **(1 pt)**

$$\frac{11!}{3!\,2!\,2!\,3!} = 277{,}200$$

b) How many permutations start and end with vowels?

Solution
Breaking this up into the different start/end vowel combinations, we have the following:
When the start and end vowel is an 'E' and 'I' pair, there are 2 arrangements of 'E' and 'I', and arranging the rest of the letters, we will need to remove the duplicates only from the two remaining 'E's, and all of the 'N's and 'G's, since there is only one 'I' left to arrange.

$$2 \cdot \frac{9!}{2!\,3!\,2!}$$

When both the start and the end are 'E's, there is only one possible unique arrangement of the start and end, and when arranging the rest of the letters there are duplicates only of 'I', 'N', and 'G' to remove, since there is only one 'E' left.

$$1 \cdot \frac{9!}{2!\,3!\,2!}$$

Finally, when both the start and the end are 'I's, there is, again, only one possible arrangement, and when arranging the rest we must consider the duplicates of 'E', 'N', and 'G'.

$$1 \cdot \frac{9!}{3!\,3!\,2!}$$

Thus, our final answer is the sum of the counts, since our three cases were disjoint. **(5 pts – 2 pts for first component, 1 pt for next two components, 1 pt for adding)**

$$2 \cdot \frac{9!}{2!\,3!\,2!} + \frac{9!}{2!\,3!\,2!} + \frac{9!}{3!\,3!\,2!} = 50{,}400$$

c) How many permutations do NOT have consecutive vowels in them?

<u>Solution</u>

This problem is equivalent to placing vowels on the lines between consonants in the string

$$|C_1|C_2| \ldots |C_6|$$

Where C_i is the i-th consonant.

Since there are (6+1)=7 places to put a vowel, 5 vowels in the string, and we want to ensure that no spot is chosen more than once, the total number of such choices is $\binom{7}{5}$

Now we need to account for rearrangements of vowels and consonants.

The total number of unique rearrangements of vowels is $\frac{5!}{3!2!}$, the number of total permutations divided by the number of duplicate ones caused by the 3 'E's and 2 'I's.

The total number of unique rearrangements of consonants is $\frac{6!}{2!3!1!}$, or the number of total permutations divided by the number of duplicates caused by the 2 'G's, 3 'N's and 1 'R'. **(4 pts total – 1 pt for each part, 1 pt for multiplying)**

Thus, our final answer is the product of these values, or

$$\binom{7}{5} \cdot \frac{5!}{3!\,2!} \cdot \frac{6!}{2!\,3!\,1!} = 12{,}600$$

d) How many permutations are the letters in alphabetical order?

<u>Solution</u>

The alphabetical ordering of the letters is as follows: E, G, I, N, R. The number of permutations where the letters are in alphabetical order will be all of the permutations which are exactly EEEGGIINNNR, which, since the letters are indistinguishable, is just 1. **(1 pt)**

e) How many permutations contain the substring "RING"?

<u>Solution</u>

There are (11-4+1) ways to place the substring "RING" (how many different starting places it can have in the string). After finding the starting place, we must choose which R, I, N, and G to use from all which are available, and then arrange the rest of the letters.

The total number of arrangements is $8 \cdot \binom{1}{1} \cdot \binom{2}{1} \cdot \binom{3}{1} \cdot \binom{2}{1} \cdot 7!$
Removing duplicates, we get

$$\frac{8 \cdot \binom{1}{1} \cdot \binom{2}{1} \cdot \binom{3}{1} \cdot \binom{2}{1} \cdot 7!}{3!\,2!\,2!\,3!} = 3{,}360$$

An alternate solution is treating "RING" as a superletter. Then we are permuting 8 letters total, with 3 Es and 2 Is, so we can just use the permutation with repetition formula to get $\frac{8!}{3!2!} = 3360$. **(2 pts)**
Note: This question gets more difficult if there are two Rs in the set of letters since two RINGs could appear.

3) (13 pts) A class contains 22 girls and 18 boys. For all parts of this question, each boy and girl are distinguishable from one another. Answer the following questions:

a) In how many ways can a committee of one boy and one girl be chosen?

Solution
Since we want exactly one girl and one boy, $\binom{18}{1} \cdot \binom{22}{1} = 18 \cdot 22 = 396$ **(2 pts)**

b) In how many ways can a committee of five students be chosen?

Solution
We want a committee of students of either gender. There are a total of 40 people, and no one person can be chosen more than once, so we have $\binom{40}{5} = 658{,}008$ ways of choosing. **(1 pt)**

c) In how many ways can a committee of two girls and three boys be chosen?

Solution
We must first choose 2 girls of the possible 22, and then choose 3 boys of the possible 18. Since for each choice of 2 girls we have to choose 3 boys we multiply these values.
So we have $\binom{22}{2} \cdot \binom{18}{3} = 188{,}496$ ways of choosing. **(2 pts)**

d) In how many ways can a committee of five students be chosen such that all the students on the committee are the same sex?

Solution
We can choose 5 students from the set of girls or 5 students from the set of boys, so we must calculate these numbers individually.
So we have $\binom{22}{5} + \binom{18}{5} = 34{,}902$ ways of choosing. **(2 pts)**

e) In how many ways can the girls and boys form a line where no two boys are standing next to one another?

Solution
Let *g* be the number of girls, and *b* be the number of boys.

Imagine lining up all of the girls (G's), leaving a placeholder (|) for a boy to potentially be placed:

$$|G_1|G_2| \dots |G_{g-1}|G_g|$$

Placing b boys in the line such that no two are standing next to each other amounts to placing these boys in these placeholders, where no one placeholder can be chosen more than once. There are $\binom{g+1}{b} = \binom{22+1}{18} = \binom{23}{18}$ such choices, as there are g+1 placeholders.

Now we must find all of the arrangements of the girls and boys. Since every person is distinguishable, there are 22! rearrangements of the girls, and 18! rearrangements of boys.

Thus there are a total of $\binom{23}{18} \cdot 22! \cdot 18! \approx 2.42 \cdot 10^{41}$ total lineups that satisfy the condition that no two boys are standing next to each other. **(3 pts – 1 pt each part)**

f) How many committees of five students contain at least two girls?

<u>Solution</u>

There are $\binom{40}{5}$ possible committees. Of these, we don't want to count the ones with zero or one girl. There are $\binom{18}{5}$ and $\binom{18}{4}\binom{22}{1}$ of these possible committees, respectively. Thus, the final result is $\binom{40}{5} - \binom{18}{5} - \binom{18}{4}\binom{22}{1} = 582{,}120$. **(3 pts – 1 pt each part)**

4) (5 pts) How many solutions does the equation a + b + c + d + e + f = 20 have if each variable must be a non-negative integer and a < 3, b < 5 and d > 4?

<u>Solution</u>

We can remove the d > 4 condition by substituting d = d'+5, which will force d to be strictly greater than 4.

$$a + b + c + d' + 5 + e + f = 20 \rightarrow a + b + c + d' + e + f = 15$$

We still need to satisfy the other two conditions, though, and we can do that by subtracting the number of solutions where the property does not hold true (found through the same method as above) from the above number.

First, substituting a = a' + 3, forcing a to be at least 3 (where the property a < 3 does not hold)

$$a' + 3 + b + c + d' + e + f = 15 \rightarrow a' + b + c + d' + e + f = 12$$

And substituting b = b' + 5

$$a + b' + 5 + c + d' + e + f = 15 \rightarrow a + b' + c + d' + e + f = 10$$

And finally, since we double counted some solutions to the equation (when both a ≥ 3 and b ≥ 5), find the number of these to correct ourselves later.

$$a' + 3 + b' + 5 + c + d' + e + f = 15 \rightarrow a + b' + c + d' + e + f = 7$$

Finally, using inclusion-exclusion, we have that the number of total solutions which satisfy the criteria is

$$\binom{15+6-1}{6-1} - \binom{12+6-1}{6-1} - \binom{10+6-1}{6-1} + \binom{7+6-1}{6-1} = \binom{20}{5} - \binom{17}{5} - \binom{15}{5} + \binom{12}{5} = 7{,}105$$

Grading: 1 pts for using basic combo with repetition, 1 pt for sub out 3s, 1 pt for sub out 5s, 1 pt for using I/E, 1 pt final answer.

5) (5 pts) How many solutions does the equation a + b + c + d + e + f + g ≤ 50 have if each variable must be a non-negative integer?

Solution

This expression can be reduced to the following problem:

Find the number of solutions to the expression a + b + c + d + e + f + g + h = 50

By adding a new term, h, we allow the sum, a + b + c + d + e + f + g, to be less than 50, and have the new term, h, pick up the slack. The original expression can still equal 50 when h is 0. This new problem is easy to solve via the standard method.

$$\binom{50+8-1}{8-1} = \binom{57}{7} = 264{,}385{,}836$$

Grading: 2 pts for extra variable, 2 pts for using some form of combos with repetition, 1 pt for final answer.

6) (5 pts) There are N users and M servers with M ≥ N. Each user can send a request to any of the servers. Determine the number of situations in which at least one collision occurs, i.e., there is at least one pair of users that send the request to the same server**.**

Solution

We want the number of ways such that there was at least one conflict. This value is equivalent to the number of ways where there was no conflict subtracted from total number of ways for requests to be sent out.

The total number of ways for n people to send a single request to one of m servers is m^n, since each of the n people can choose any of the m servers.

The number of ways for n people to send a single request to one of m servers where there are no collisions would be P(m, n), since this is the same as finding an arrangement of length n of m servers.

So, the number of ways for m people to send requests to n servers where there was at least one conflict is $m^n - P(m,n)$ **(2 pts for each part, 1 pt for subtraction)**

7) (5 pts) How many integers in between 1 and 10^7 are divisible by 3, 5 or 7?

Solution

Note that IE for 3 sets is:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

The number of positive numbers below 10^7 which are divisible by n (including n) is $\left\lfloor \frac{10^7}{n} \right\rfloor$, so our total is:

The number divisible by 3 + divisible by 5 + divisible by 7 – divisible by 3 and 5 – divisible by 3 and 7 – divisible by 5 and 7 + divisible by 3 and 5 and 7

$$\left\lfloor \frac{10^7}{3} \right\rfloor + \left\lfloor \frac{10^7}{5} \right\rfloor + \left\lfloor \frac{10^7}{7} \right\rfloor - \left\lfloor \frac{10^7}{3 \cdot 5} \right\rfloor - \left\lfloor \frac{10^7}{3 \cdot 7} \right\rfloor - \left\lfloor \frac{10^7}{5 \cdot 7} \right\rfloor + \left\lfloor \frac{10^7}{3 \cdot 5 \cdot 7} \right\rfloor = 5{,}428{,}572$$

Grading: 2 pts for floor formula, 2 pts for I/E, 1 pt for result