COT 3100 Homework #7 Solutions (by Arup Guha)
Mathematical Induction

1) Use mathematical induction on n to prove that $\sum_{i=1}^{n}\frac{1}{i^2} < 2 - \frac{1}{n}$, for all integers n > 1.

Solution

Base case: n = 2 LHS = $\sum_{i=1}^{2}\frac{1}{i^2} = 1 + \frac{1}{4} = \frac{5}{4}$, RHS = $2 - \frac{1}{2} = \frac{3}{2}$, since $\frac{5}{4} < \frac{3}{2}$, the assertion holds for n = 2. **(1 pt)**

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k, greater than 1, that $\sum_{i=1}^{k}\frac{1}{i^2} < 2 - \frac{1}{k}$. **(1 pt)**

Inductive step: Prove for n = k+1 that $\sum_{i=1}^{k+1}\frac{1}{i^2} < 2 - \frac{1}{k+1}$. **(1 pt)**

$\sum_{1}^{k+1}\frac{1}{i^2} = \sum_{1}^{k}\frac{1}{i^2} + \frac{1}{(k+1)^2}$ **(1 pt)**

$< \sum_{1}^{k}\frac{1}{i^2} + \frac{1}{k(k+1)}$, since k+1 > k and k > 1, it follows that $\frac{1}{k+1} < \frac{1}{k}$. **(2 pts)**

$< 2 - \frac{1}{k} + \frac{1}{k(k+1)}$, using the inductive hypothesis. **(2 pts)**

$= 2 - \frac{(k+1)-1}{k(k+1)}$ **(1 pt)**

$= 2 - \frac{k}{k(k+1)}$

$= 2 - \frac{1}{(k+1)}$ **(1 pt)**

This completes the inductive step. Thus we can conclude for all positive integers n > 1 that $\sum_{i=1}^{n}\frac{1}{i^2} < 2 - \frac{1}{n}$.

2) Use mathematical induction on n to prove that $\sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{(n-1)n}{4}$, for all positive integers n. Note that $H_n = \sum_{i=1}^{n}\frac{1}{i}$.

Solution

Base case: n = 1: LHS = $\sum_{i=1}^{n} iH_i = H_1 = 1$, RHS = $\frac{1(1+1)}{2}H_1 - \frac{(1-1)1}{4} = H_1 = 1$, so the formula holds for n = 1. **(1 pt)**

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that $\sum_{i=1}^{k} iH_i = \frac{k(k+1)}{2}H_k - \frac{(k-1)k}{4}$. **(1 pt)**

Inductive step: Prove for n = k+1 that $\sum_{i=1}^{k+1} iH_i = \frac{(k+1)(k+2)}{2}H_{k+1} - \frac{k(k+1)}{4}$. **(1 pt)**

$\sum_{i=1}^{k+1} iH_i = \left(\sum_{i=1}^{k} iH_i\right) + (k+1)H_{k+1}$, splitting off the last term **(1 pt)**

= $\frac{k(k+1)}{2}H_k - \frac{(k-1)k}{4} + (k+1)H_{k+1}$, using the I.H. **(1 pt)**

= $\frac{k(k+1)}{2}(H_{k+1} - \frac{1}{k+1}) - \frac{(k-1)k}{4} + (k+1)H_{k+1}$, since $H_{k+1} = H_k + \frac{1}{k+1}$* **(1 pt)**

= $\frac{k(k+1)}{2}H_{k+1} - \frac{k}{2} - \frac{(k-1)k}{4} + (k+1)H_{k+1}$ **(1 pt)**

= $(k+1)H_{k+1}\left(\frac{k}{2}+1\right) - (\frac{2k}{4} + \frac{(k-1)k}{4})$ **(1 pt)**

= $(k+1)H_{k+1}\left(\frac{k+2}{2}\right) - (\frac{k^2-k+2k}{4})$ **(1 pt)**

= $\left(\frac{(k+1)(k+2)}{2}\right)H_{k+1} - (\frac{k^2+k}{4})$

= $\left(\frac{(k+1)(k+2)}{2}\right)H_{k+1} - (\frac{k(k+1)}{4})$ **(1 pt)**

This completes the inductive step. Thus, we have shown for all positive integers n, $\sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{(n-1)n}{4}$.

*Note: $H_{k+1} = \sum_{i=1}^{k+1}\frac{1}{i} = (\sum_{i=1}^{k}\frac{1}{i}) + \frac{1}{k+1} = H_k + \frac{1}{k+1}$.

3) Define a sequence of numbers, W, as follows: $W_0 = 2$, $W_1 = 1$ and $W_n = W_{n-1} + \frac{W_{n-2}}{2}$ for all integers *n* > 1. For all positive integers n, prove that $\sum_{i=1}^{n} \frac{W_{i-1}}{W_i W_{i+1}} = 2 - \frac{2}{W_{n+1}}$.

Solution

Before we start solving the problem, it'll be important to rewrite our given recurrence $W_n = W_{n-1} + \frac{W_{n-2}}{2}$ as $W_{n-1} = W_n - \frac{W_{n-2}}{2}$, which is equivalent to $2W_{n-1} = 2W_n - W_{n-2}$. We will use a form of this substitution below.

Base case: n = 1: LHS = $\sum_{i=1}^{n} \frac{W_{i-1}}{W_i W_{i+1}} = \frac{W_0}{W_1 W_2} = \frac{2}{1(2)} = 1$,RHS = $2 - \frac{2}{W_{n+1}} = 2 - \frac{2}{W_2} = 2 - 1 = 1$, thus the formula holds for n = 1. **(1 pt)**

Inductive hypothesis: Assume for an arbitrarily chosen n = k that $\sum_{i=1}^{k} \frac{W_{i-1}}{W_i W_{i+1}} = 2 - \frac{2}{W_{k+1}}$. **(1 pt)**

Inductive step: Prove for n = k+1 that $\sum_{i=1}^{k+1} \frac{W_{i-1}}{W_i W_{i+1}} = 2 - \frac{2}{W_{k+2}}$. **(1 pt)**

$\sum_{i=1}^{k+1} \frac{W_{i-1}}{W_i W_{i+1}} = \left(\sum_{i=1}^{k} \frac{W_{i-1}}{W_i W_{i+1}}\right) + \frac{W_k}{W_{k+1} W_{k+2}}$, splitting off the first term. **(1 pt)**

$= 2 - \frac{2}{W_{k+1}} + \frac{W_k}{W_{k+1} W_{k+2}}$, using the inductive hypothesis. **(1 pt)**

$= 2 - \frac{2W_{k+2}}{W_{k+1} W_{k+2}} + \frac{W_k}{W_{k+1} W_{k+2}}$ **(1 pt)**

$= 2 - \frac{2W_{k+2} - W_k}{W_{k+1} W_{k+2}}$ **(1 pt)**

$= 2 - \frac{2W_{k+1}}{W_{k+1} W_{k+2}}$, using adjusted recurrence developed at the beginning. **(2 pts)**

$= 2 - \frac{2}{W_{k+2}}$ **(1 pt)**

This completes the inductive step. Thus, we can conclude for all positive integers n, $W_n = W_{n-1} + \frac{W_{n-2}}{2}$.

4) Whenever Binary Billy acts up, his punishment is to write binary numbers on the board. He always starts writing 0, 1, 10, 11, 100, etc. Depending on the severity of behavior, Billy has to write all the binary numbers starting at 0 upto all binary numbers with a certain number of digits. For example, if Billy's bad behavior was rated at a 5, then Billy would have to write all the binary numbers from 0 through 11111. Let B(n) denote the total number of binary *digits* Billy must write for a bad behavior rating of n. Using induction on n, prove that B(n) = (n-1)2^n + 2, for all positive integers n.

Solution

Base case: n=1. Billy must write two digits, 0 and 1, thus B(1) = 2. Looking at the right hand side we find it equal to (1-1)2^1 + 2 = 2, thus the given formula is true for n=1. **(1 pt)**

Inductive hypothesis: Assume that for an arbitrary value of n=k that $B(k) = (k-1)2^k + 2$. **(1 pt)**

Inductive step: Prove, under the inductive hypothesis, that for n=k+1, $B(k+1) = k2^{k+1} + 2$. **(1 pt)**

Let b(n) = number of binary numbers with EXACTLY n digits. For example, b(2) = 2 since there are two binary numbers, 10 and 11 with exactly 2 digits.

In particular, using the multiplication principle, we have that b(n) = 2^{n-1}. This is because the first binary digit of the n digits must be 1. The rest can either be 0 or 1. Thus, we have 2 choices for each of the n-1 remaining digits. Since each of these choices are independent, we multiply 2 by itself n-1 times to obtain the total number of possible binary numbers with exactly n digits. **(1 pt)**

Now, to prove the assertion:

$B(k+1) = B(k) + (k+1)b(k+1)$, since we are adding b(k+1) values with k+1 digits. **(1 pt)**
$= (k-1)2^k + 2 + (k+1)2^k$, using the inductive hypothesis. **(2 pts)**
$= (k-1+k+1)2^k + 2$ **(1 pt)**
$= (2k)2^k + 2$ **(1 pt)**
$= k2^{k+1} + 2$ **(1 pt)**

This completes the inductive step. Thus, we can assume for all positive integers n that $B(n) = (n-1)2^n + 2$.

5) There are n cars on a circular track, and amongst them there is enough gas for 1 car to make a complete loop around the track. Show that there is 1 car that can make it completely around the track by pooling gas from every car that it passes by. Though other techniques than mathematical induction can be used to prove this, please use induction to do so. If you can't come up with an inductive proof, but can provide another one, please do so for partial credit.

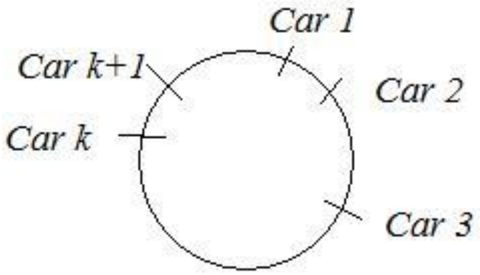
Base case: n = 1. If we have one car on a track and it has enough gas to go around the track, then, without any pooling, so to speak, it can go all the way around the track, proving the base case. **(1 pt)**

Inductive hypothesis: Assume for an arbitrary positive integer n = k that if there are k cars around a track with a sum of gasoline equivalent to making it around the track, that at least one of the cars is in a position that it can make it around the track by pooling gas from all of the other cars (which, of course, remain stationary =)). **(1 pt)**

Inductive step: Prove for n = k + 1 that if there are k+1 cars around a track with a sum of gasoline equivalent to making it around the track, that at least one of the cars is in a position that it can make it around the track by pooling gas from all of the other cars. **(1 pt)**

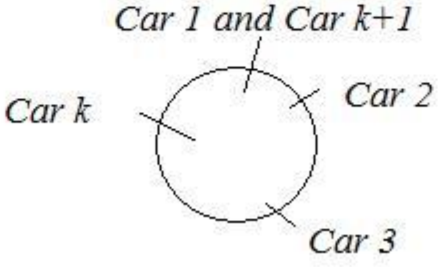
Solution

Consider an arbitrary track with k+1 cars pictured below:



Since the cars in total have enough gas to go around the track, it follows that at least one of the cars has enough gas to make it to the next car. (If EVERY car did not have enough gas to make it to the original spot of the next car, than the sum of distances all the cars could drive with their gas would be definitively LESS THAN the length of the track, contradicting the original premise.) Without loss of generality, let this car be car k+1. (Once we find our car that satisfies this constraint, we can always renumber our cars around the circle to fit this convention…) **(2 pts for observation that one car has enough gas to get to the next.)**

Now, consider cutting out the segment of the track from car k+1 to car 1. Also, consider giving car 1 the amount of excess gas that car k+1 has. Remember, that we assumed that car k+1 has at least as much gas as needed to get to car 1. But, it may have some excess gas. Whatever this excess is, give it to car 1. In doing so, we make sure that the amount of gas cars 1 through k has is exactly equal to the amount necessary to go around a shortened track with car k+1's segment cut out. Our picture looks like this: **(2 pts for trying to cut that car out and create a situation for which the IH applies)**



Our inductive hypothesis applies to this "subtrack" of our inductive step. Thus, at least one of the cars in this picture can make it all the way around the track, pooling gas from the other cars. For visualization purposes, assume that the car we choose travels clockwise around the track. Now, imagine this car in our original track for the inductive step. The only difference will be that when our chosen car gets to the label "Car 1 and Car k+1" in the shortened track, it'll really be meeting up with car k+1 in the original track and be able to pool ALL of its gas. Thus, it'll have enough gas to make it to car 1 and potentially more. Thus, since our car can make it around the shortened track, the extra supply from car k+1 will precisely give it what it needs to make up the extra track in between car k+1 and car 1. From there, we know it'll make it all the way around because in our I.H. that excess gas we gave car 1 will be gas that our chosen car already got from car k+1 in our original track. **(3 pts for this whole explanation, there's quite a bit going on here.)**

This proves the inductive step. As a consequence, we can conclude that for any number of cars on a track that have enough gas collectively to circle the track, there will exist at least one car that can make it around the track, pooling gas from the rest.

Grading note: If an alternate non-inductive proof is given, give a grade of 7 max. There are many ways to do this. Most of them hinge on the fact that in any given interval a function has a minimum value. (Basically, imagine a real valued function that starts at any car and subtracts gas out as the car drives and "steps up" whenever it gets the gas of a subsequent car. This function will start at f(0) = c where c > 0 and end with f(1) = 0, where we assume the length of the track to be 1. This function may dip negative, which means that in real life the car pooling gas would run out and have to "borrow" gas… But, imagine going to the minimum point on this function. Then, imagine redoing the function so that this point represented f(0). You'll notice from here, due to the cyclic nature of the function, it's impossible for it to dip below zero.)