Spring 2014 COT 3100 Homework #6 Solutions (by Ryan Zimmerman)
Mathematical Induction

1) Use mathematical induction to prove that $\sum_{i=1}^{n}(i \times i!) = (n+1)! - 1$, for all positive integers n.

Base case: Let n=1:

$LHS = \sum_{i=1}^{1}(i \times i!) = 1 \times 1! = 1$, $RHS = (1+1)! - 1 = 1$.

Thus, the given statement is true for n = 1. **(1 pt)**

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k: **(1 pt)**

$\sum_{i=1}^{k}(i \times i!) = (k+1)! - 1$

Inductive step: Prove for n = k+1: **(2 pts)**

$$\sum_{i=1}^{k+1}(i \times i!) = (k+2)! - 1$$

$\sum_{i=1}^{k+1}(i \times i!) = \left[\sum_{i=1}^{k}(1 \times i!)\right] + (k+1)(k+1)!$, **(1 pt)**

$= (k+1)! - 1 + (k+1)(k+1)!$, using IH **(2 pts)**
$= (k+1)! - 1 + (k+2)! - (k+1)!$, (See Note Below) **(2 pts)**
$= (k+2)! - 1$, completing the inductive step. **(1 pt)**

Therefore $\sum_{i=1}^{n}(i \times i!) = (n+1)! - 1$, for all positive integers n, by mathematical induction.

QED

Note:

$$\begin{aligned}(k+1)(k+1)! &= \big((k+2) - 1\big)(k+1)! \\ &= (k+2)(k+1)! - (k+1)! \\ &= (k+2)! - (k+1)!\end{aligned}$$

2) Use mathematical induction to prove that $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$, for all positive integers n, where F_k represents the k^{th} Fibonacci number.

Base case: Let n=1:

$LHS = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $RHS = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, so the given formula is true for n = 1. **(1 pt)**

Inductive hypothesis: Assume for an arbitrary positive integer n = k that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} \textbf{ (1 pt)}$$

Inductive step: Prove for n = k+1 that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k+1} = \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix} \textbf{ (2 pts)}$$

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k+1} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} && \textbf{(1 pt)} \\ &= \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \text{ using the IH} && \textbf{(2 pts)} \\ &= \begin{bmatrix} F_{k+1} + F_k & F_{k+1} \\ F_k + F_{k-1} & F_k \end{bmatrix} && \textbf{(1 pt)} \\ &= \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix}, \text{ using the Fibonacci identity} && \textbf{(2 pts)} \end{aligned}$$

It follows that for all positive integers n, $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$.

QED

3) For all positive integers n, use mathematical induction to prove that $21 \mid (4^{n+1} + 5^{2n-1})$.

Base case: n = 1

Plugging in n = 1 to our expression, we get $4^2 + 5^1 = 21$. $21 \mid 21$ ✓ **(1pt)**

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$21 \mid (4^{k+1} + 5^{2k-1})$$

Thus, there exists an integer c such that $4^{k+1} + 5^{2k-1} = 21c$. **(1 pt)**

Inductive step: Prove for n = k+1: $21 \mid (4^{k+2} + 5^{2k+1})$. **(1 pt)**

$4^{k+2} + 5^{2k+1} = 4*4^{k+1} + 5^2*5^{2k-1}$ **(1 pt)**
$= 4*4^{k+1} + ((4+21)*5^{2k-1})$ **(2 pts)**
$= 4*(4^{k+1} + 5^{2k-1}) + 21*5^{2k-1}$ **(1 pt)**
$= 4(21c) + 21(5^{2k-1})$, using IH **(2 pts)**
$= 21(4c + 5^{2k-1})$ **(1 pt)**

Since c is an integer and k is a positive integer, it follows that $4c + 5^{2k-1}$ is integral. Thus, $21 \mid (4^{k+2} + 5^{2k+1})$ as desired.✓

Therefore $21 \mid (4^{n+1} + 5^{2n-1})$ for all positive integers n, by mathematical induction

QED

4) Use mathematical induction to prove that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$, for all positive integers n. Note: The trigonometry formulas you may find useful are as follows:

$\cos(A + B) = (\cos A)(\cos B) - (\sin A)(\sin B)$
$\sin(A + B) = (\sin A)(\cos B) + (\cos A)(\sin B)$

This statement is true more generally (for non-integral values of n also) and is known as DeMoivre's Theorem.

Base case: $n = 1$

LHS $= (\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$, RHS $= \cos(1\theta) + i\sin(1\theta) = \cos\theta + i\sin\theta$, showing that the formula is true for $n = 1$. **(1 pt)**

Inductive hypothesis: Assume ofr an arbitrary positive integer $n = k$ that

$$(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta).$$ **(1 pt)**

Inductive step: Prove for $n = k+1$ that

$$(\cos\theta + i\sin\theta)^{k+1} = \cos((k+1)\theta) + i\sin((k+1)\theta).$$ **(1 pt)**

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$$ **(1 pt)**
$$= (\cos(k\theta) + i\sin(k\theta))(\cos\theta + i\sin\theta)$$ **(2 pts)**
$$= (\cos(k\theta)\cos\theta) + i(\cos(k\theta)\sin\theta) + i(\sin(k\theta)\cos\theta) + i^2(\sin(k\theta)\sin\theta)$$
$$= (\cos(k\theta)\cos\theta) - (\sin(k\theta)\sin\theta) + i(\cos(k\theta)\sin\theta + \sin(k\theta)\cos\theta)$$
$$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$$
$$= \cos((k+1)\theta) + i\sin((k+1)\theta)$$

It follows that DeMoivre's Theorem is true for all positive integers n.

Grading cont: FOIL = 1 pt, combine 1 pt, 1 pt use identity, 1 pt simplify

5) Use mathematical induction to prove that $\sum_{i=1}^{n}(i(i+1)(i+2)) = \frac{n(n+1)(n+2)(n+3)}{4}$, for all positive integers n.

Base Case: n = 1

$LHS = \sum_{i=1}^{1}(i(i+1)(i+2)) = 1(2)(3) = 6, RHS = \frac{1(1+1)(1+2)(1+3)}{4} = 6$
Thus, the given equation is true for n = 1. **(1 pt)**

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that **(1 pt)**

$$\sum_{i=1}^{k}(i(i+1)(i+2)) = \frac{k(k+1)(k+2)(k+3)}{4}.$$

Inductive step: Prove for n = k+1 that **(2 pts)**

$$\sum_{i=1}^{k+1}(i(i+1)(i+2)) = \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

$\sum_{i=1}^{k+1}(i(i+1)(i+2)) = \left[\sum_{i=1}^{k}\left(i(i+1)(i+2)\right)\right] + (k+1)(k+2)(k+3)$ **(1 pt)**

$= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$, using IH **(2 pts)**
$= \frac{k(k+1)(k+2)(k+3)}{4} + \frac{4(k+1)(k+2)(k+3)}{4}$ **(2 pts)**
$= \frac{[(k+1)(k+2)(k+3)](k+4)}{4}$
$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$, as desired. **(1 pt)**

It follows that that $\sum_{i=1}^{n}(i(i+1)(i+2)) = \frac{n(n+1)(n+2)(n+3)}{4}$, for all positive integers n.

6) (Extra Credit) The sum shown in #5 is a specific instance of the more general following result:

$$\sum_{i=1}^{n}(\prod_{j=i}^{i+k} j) = \frac{\prod_{i=n}^{n+k+1} i}{k+2}$$

for all positive integers n and non-negative integers k. Use induction on n to prove this result. (In doing so, treat k as an arbitrary fixed non-negative integer.)

Base case: n = 1

$LHS = \sum_{i=1}^{1}(\prod_{j=i}^{i+k} j) = (k+1)!$, $RHS = \frac{\prod_{i=1}^{1+k+1} i}{k+2} = \frac{(k+2)!}{k+2} = (k+1)!$, thus, the given formula is true for n = 1. **(1 pt)**

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = m that: **(1 pt)**

$$\sum_{i=1}^{m}(\prod_{j=i}^{i+k} j) = \frac{\prod_{i=m}^{m+k+1} i}{k+2}$$

Inductive step: Prove for n = m+1: **(1 pt)**

$$\sum_{i=1}^{m+1}(\prod_{j=i}^{i+k} j) = \frac{\prod_{i=m+1}^{m+1+k+1} i}{k+2}$$

$$\begin{aligned}
\sum_{i=1}^{m+1}(\prod_{j=i}^{i+k} j) &= [\sum_{i=1}^{m}(\prod_{j=i}^{i+k} j)] + \prod_{j=m+1}^{m+1+k} j \\
&= \frac{\prod_{i=m}^{m+k+1} i}{k+2} + \prod_{j=m+1}^{m+1+k} j \\
&= \frac{(m+k+1)!}{(m-1)!\,(k+2)} + \frac{(m+1+k)!}{m!} \\
&= \frac{(m+k+1)!}{(m-1)!}[\frac{1}{k+2} + \frac{1}{m}] \\
&= \frac{(m+k+1)!}{(m-1)!}[\frac{m+(k+2)}{m(k+2)}] \\
&= \frac{(m+k+2)!}{m!\,(k+2)} \\
&= \frac{\prod_{i=m+1}^{m+1+k+1} i}{k+2}
\end{aligned}$$

Grading: 1 pt split, 2 pts IH, 1 pt write !, 1 pt factor, 1 pt common, 1 pt finish