Spring 2014 COT 3100 HW4 Solutions (by Ivette Carreras)

1) (5 pts) An arithmetic sequence of 40 terms $a_1, a_2, \ldots, a_{40}$ has $a_{17} = 25$ and $a_{33} = 121$. Find the sum of the sequence.

Solution

$$a_k - a_j = (k - j) * d$$

$$\frac{a_{33} - a_{17}}{(33 - 17)} = d$$

Find the difference: $d = \frac{96}{16} = 6$ **(1 pt)**

$$a_{17} - a_1 = (17 - 1) * 6$$

Find a_1: $a_1 = -((16 * 6) - 25) = -71$ **(1 pt)**

Find a_n , n = 40: $a_{40} = a_1 + (40 - 1) * 6 = 163$ **(1 pt)**

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$S_{40} = \frac{40}{2}(-71 + 163) = 1840$ **(2 pts)**

2) (5 pts) A geometric sequence of 20 terms terms $a_1, a_2, \ldots, a_{20}$ has $a_{12} = 42$ and $a_{18} = 336$. Find the sum of the sequence.

Solution

$$\frac{a_j}{a_k} = r^{j-k}$$

Find the ratio r: $r^{18-12} = \frac{a_{18}}{a_{12}} = \frac{336}{42}$

$r = \sqrt{2}$ **(1 pt)**

Find a_1: $\frac{a_{12}}{a_1} = r^{12-1} = \sqrt{2}^{11}$

$a_1 = \frac{42}{\sqrt{2}^{11}} = \frac{21}{32} * \sqrt{2}$ **(2 pts)**

$S_n = a_1 * \frac{(1-r^n)}{(1-r)}$, so $S_{20} = \frac{21}{32} * \sqrt{2} * \frac{\left(1-\sqrt{2}^{20}\right)}{(1-\sqrt{2})}$ **(2 pts)**

3) (10 pts) Let S = $\sum_{i=1}^{\infty}(2i-1)(\frac{2}{3})^i$. Write out the first five terms of S. Use either the subtraction technique or derivative technique shown in class to find the value of S.

Solution

Let a_i denote the i^{th} term in the sequence. We have

$a_1 = \mathbf{1} * \left(\frac{2}{3}\right)^{\mathbf{1}}$, $a_2 = \mathbf{3} * \left(\frac{2}{3}\right)^{\mathbf{2}}$, $a_3 = \mathbf{5} * \left(\frac{2}{3}\right)^{\mathbf{3}}$, $a_4 = \mathbf{7} * \left(\frac{2}{3}\right)^{\mathbf{4}}$, $a_5 = 9 * \left(\frac{2}{3}\right)^{\mathbf{5}}$

$$\boldsymbol{S} = \mathbf{1} * \left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{1}} + \mathbf{3} * \left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{2}} + \mathbf{5} * \left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{3}} + \mathbf{7} * \left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{4}} + \cdots$$

$$-\frac{\mathbf{2}}{\mathbf{3}}\boldsymbol{S} = \qquad \mathbf{1} * \left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{2}} + \mathbf{3} * \left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{3}} + \mathbf{5} * \left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{4}} + \mathbf{7} * \left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{5}} + \cdots$$

$$\frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{S} = \frac{\mathbf{2}}{\mathbf{3}} + \mathbf{2} * \left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{2}} + \mathbf{2} * \left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{3}} + \mathbf{2} * \left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{4}} + \mathbf{2} * \left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{5}} + \cdots$$

$$\frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{S} = \frac{\mathbf{2}}{\mathbf{3}} + \left[\frac{\mathbf{8}}{\mathbf{9}}\left(\mathbf{1} + \left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{1}} + \left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{2}} + \left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{3}} + \cdots\right)\right]$$

$$\boldsymbol{S} = \mathbf{2} + \left[\frac{\mathbf{8}}{\mathbf{3}} * \sum_{\boldsymbol{i}=\mathbf{0}}^{\infty}\left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\boldsymbol{i}}\right]$$

$$\boldsymbol{S} = \mathbf{2} + \frac{\mathbf{8}}{\mathbf{3}}(\mathbf{3}) = \mathbf{10}$$

Grading: Writing 5 terms = 2pts

Writing sequence out = 1 pt

Writing sequence * 2/3 = 1 pts

Subtracting = 2 pts

Identifying Geo Sum in subtraction = 1 pt

Solving Geo sum = 2 pts

Final answer = 1pt

4) (10 pts) Let T = $\sum_{i=1}^{20}(2i-1)(\frac{2}{3})^i$. (Essentially, T is the first 20 terms of S.) Use the work in your solution for #3 to determine T. Do not solve for a decimal value of T. Instead, given an expression for T in terms of the various constants involved in the problem.

Solution

Using the work from question 3, but instead of going to infinity now we get the first 20 elements:

$$\boldsymbol{S = 1*\left(\frac{2}{3}\right)^1 + 3*\left(\frac{2}{3}\right)^2 + 5*\left(\frac{2}{3}\right)^3 + 7*\left(\frac{2}{3}\right)^4 + \cdots + 39(\frac{2}{3})^{20}}$$

$$\boldsymbol{-\frac{2}{3}S = \qquad 1*\left(\frac{2}{3}\right)^2 + 3*\left(\frac{2}{3}\right)^3 + 5*\left(\frac{2}{3}\right)^4 + \ldots + 37(\frac{2}{3})^{20} + 39(\frac{2}{3})^{21}}$$

$$\boldsymbol{\frac{1}{3}S = \frac{2}{3} + \left(2*\left(\frac{2}{3}\right)^2 + 2*\left(\frac{2}{3}\right)^3 + 2*\left(\frac{2}{3}\right)^4 + \cdots + 2*\left(\frac{2}{3}\right)^{20}\right) - 39(\frac{2}{3})^{21}}$$

$$\boldsymbol{\frac{1}{3}S = \frac{2}{3} + [\frac{8}{9}\left(1 + \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^{18}\right] - 39(\frac{2}{3})^{21}}$$

$$\boldsymbol{S = 2 + \frac{8}{3}*\frac{\left(1-(\frac{2}{3})^{19}\right)}{\left(1-\frac{2}{3}\right)} - 39(\frac{2}{3})^{21}}$$

$$\boldsymbol{S = 2 + \frac{8}{3}*\frac{\left(1-(\frac{2}{3})^{19}\right)}{\frac{1}{3}} - 39(\frac{2}{3})^{21}}$$

$$\boldsymbol{S = 2 + 8(1 - (\frac{2}{3})^{19}) - 39(\frac{2}{3})^{21}}$$

$$\boldsymbol{S = 10 - 8(\frac{2}{3})^{19} - 39(\frac{2}{3})^{21}}$$

$$\boldsymbol{S = 10 - \left(\frac{2}{3}\right)^{19}(8 + 39\left(\frac{2}{3}\right)^2)}$$

$$\boldsymbol{S = 10 - \frac{76}{3}\left(\frac{2}{3}\right)^{19}}$$

Grading: 3 pts for invoking previous work, 3 pts for having last negative term involved, 2 pts for solving the finite geometric sum, 2 pts for the final solution (can be in different forms)

5) (10 pts) Consider the following recurrence relation: $a_1 = 1$, $a_2 = 11$, $a_n = a_{n-1} + 2a_{n-2}$, for all integers $n > 2$. Prove that the sequence $\{b_n\}$, where $b_n = 2^{n+1} + 3(-1)^n$ satisfies the given recurrence relation. Also show that $b_1 = a_1$ and $b_2 = a_2$. What conclusion can you draw from both of these observations?

Solution

$\boldsymbol{b_1 = 2^{1+1} + 3 * (-1)^1 = 1}$ **(1 pt)**
$\boldsymbol{b_2 = 2^{2+1} + 3 * (-1)^2 = 11}$ **(1 pt)**
$\boldsymbol{b_1 = a_1}$,
$\boldsymbol{b_2 = a_2}$

Now we aim to show that $b_n = b_{n-1} + 2a_{n-2}$. To show equality, we must simply show that the two sides are equal.

Note that $\boldsymbol{b_{n-1} = 2^n + 3 * (-1)^{n-1}}$ and $\boldsymbol{b_{n-2} = 2^{n-1} + 3 * (-1)^{n-2}}$

LHS = $\boldsymbol{b_{n-1} + 2 * b_{n-2}}$ **(1 pt)**

$= \boldsymbol{2^n + 3 * (-1)^{n-1} + 2 * [2^{n-1} + 3} * (-1)^{n-2}(-1)^2]$, since $(-1)^2 = 1$.

$= \boldsymbol{2^n + 2 * 2^{n-1} + 6 * (-1)^n + 3 * (-1)^{n-1}}$ **(2 pts prev, 1 pt this line)**

$= \boldsymbol{2(2^n) + 6 * (-1)^n + 3 * \frac{(-1)^n}{(-1)}}$ **(1 pt)**

$= \boldsymbol{2^{n+1} + 6 * (-1)^n - 3 * (-1)^n}$ **(1 pt)**

$= \boldsymbol{2^{n+1} + 3 * (-1)^n}$ **(1 pt)**

$= \boldsymbol{b_n}$ **(1 pt)**

Since $a_1 = b_1$, $a_2 = b_2$, and b_n follows the recurrence of a_n , we can conclude that $a_n = b_n$, for all positive integers n.

6) (5 pts) Let F_n denote the n^{th} Fibonacci number. Prove that $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$. Given this result, intuitively explain why $\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution

$$\boldsymbol{F_n = F_{n-1} + F_{n-2}}$$

$$\boldsymbol{F_{n+1} = F_n + F_{n-1}}$$

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} * \begin{bmatrix} \boldsymbol{F_n} \\ \boldsymbol{F_{n-1}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F_n + F_{n-1}} \\ \boldsymbol{F_n} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F_{n+1}} \\ \boldsymbol{F_n} \end{bmatrix}$$

From the previous multiplication we can see that if we multiply a matrix 2x1, containing two consecutive Fibonacci numbers by the matrix $\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$ we will get a 2x1 matrix with the previous Fibonacci numbers of the initial values. If we do this many times, we will reach the base cases, F_1 = 1and F_2 = 1. In the example given the first value is Fn, if we multiply $\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$ (n-2) times we will reach the base cases.

F_n back (n-2) times => F_{n-n+2} => F_2

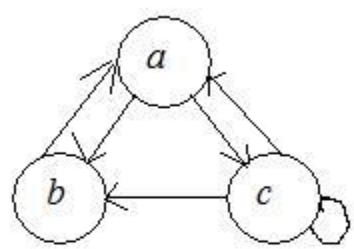
F_{n-1} back (n-2) times => $F_{n-1-n+2}$ => F_1

Grading: 2 pts for initial multiplication, 1 pt for simplifying $\boldsymbol{F_n + F_{n-1}}$, 2 pts for the intuitive justification of the final result, use your discretion.

7) (5 pts) Every company makes peculiar restrictions for passwords, but employees of ABC have particularly strange restrictions. Their passwords may only consist of the lowercase letters a, b and c. Furthermore, for any a in a password, it must be followed by a b or c. For any b in a password, it must be followed by an a, and c may be followed by any of the three letters. For examples, abaccba is a valid password of length 7. (Note that for both b's, a comes right after them.) Using a transition matrix, find the number of valid passwords of length 12. You may either use a computer program or a calculator to do the appropriate matrix exponentiation and multiplication. (Extra credit for anyone who can personally email me the full solution at dmarino@cs.ucf.edu before I go over how to solve this question in class.) Note: the work for this question is more difficult than the others but is only worth 5 points because I wanted to pose an interesting question without students' grades suffering too much.

Solution

We can model this question as a graph question. In particular, imagine a graph with three vertices, a, b and c, with transitions based on which letters are allowed to follow which letters. For examples, since b or c can follow a, we can put a directed edge from vertex a to both vertices b and c. From vertex c, we would have edges to all three vertices a, b and c. Here is the graph:



We essentially seek the number of paths in this graph, starting at a, b or c and ending at a, b or c, of length 11. The reason we are looking for paths of length 11 is that a path of length one represents a string of length 2, since where we start indicates a letter itself. **(Graph picture/description = 2 pts)**

The adjacency matrix of this graph is as follows: $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

Thus, we seek the sum of the entries of $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{11}$. Using a calculator (or computer program, we find:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{11} = \begin{bmatrix} 682 & 683 & 683 \\ 342 & 341 & 341 \\ 1024 & 1024 & 1024 \end{bmatrix}.$$

(Grading – matrix expression, many different ones work, is 2 pts)

Summing all of the entries in this matrix we get the desired total of 6144 valid passwords. **(Figuring out which items to add and the correct answer is 1 pt)**