

**Spring 2014 COT 3100 Homework #2 Solutions (by Max Miller)**

1) **Total = 10 pts**

a) The set contains perfect powers of 3, with the exponent being each integer ranging from 0 to 9, inclusive.

$$\{1, 3, 9, 27, 81, 243, 729, 2187, 6561, 19683\} \quad \text{Grading: 2 pts}$$

b) Set up the given equation, solving for n:

$$\begin{aligned} 4^n - 72(2)^n + 512 &= 0 \\ (2^n)^2 - 72(2)^n + 512 &= 0 \\ \frac{72 \pm \sqrt{(72^2 - 4 * 1 * 512)}}{2} & \\ (2^n - 8)(2^n - 64) &= 0 \\ 2^n = 8, n = 3 & \\ 2^n = 64, n = 6 & \end{aligned}$$

$$\{3, 6\} \quad \text{Grading: 4 pts - 2 for setting up equation, 1 for factoring, 1 for solutions}$$

c) Plug in the given integers for n in the function specified to get the desired set.

$$\begin{aligned} 0^3 - 4 * 0^2 + 3 * 0 &= 0 \\ 1^3 - 4 * 1^2 + 3 * 1 &= 0 \\ 2^3 - 4 * 2^2 + 3 * 2 &= -2 \\ 3^3 - 4 * 3^2 + 3 * 3 &= 0 \\ 4^3 - 4 * 4^2 + 3 * 4 &= 12 \end{aligned}$$

$$\{-2, 0, 12\} \quad \text{Grading: 2 pts}$$

d) The original question was a typo. The set A is simply the set of non-negative integers divisible by 4 that are less than 20.

$$\{0, 4, 8, 12, 16\} \quad \text{Grading: 2 pts}$$

2) **Total = 15 pts, 3 pts per part**

a) Here, we are looking for all items not in any of the sets. (**Grading: 2 pts for subtraction, 1 answer**)

$$\begin{aligned} |U - ((A \cup B) \cup C)| \\ |U| - |(A \cup B) \cup C \cap U| \\ |U| - |(A \cup B) \cup C| \\ 50 - 40 = 10 \end{aligned}$$

b) Once again, we can directly subtract two of the given items to find the number of items that don't belong to all three sets. (**Grading: 2 pts for subtraction, 1 answer**)

$$|U - (A \cap (B \cap C))|$$

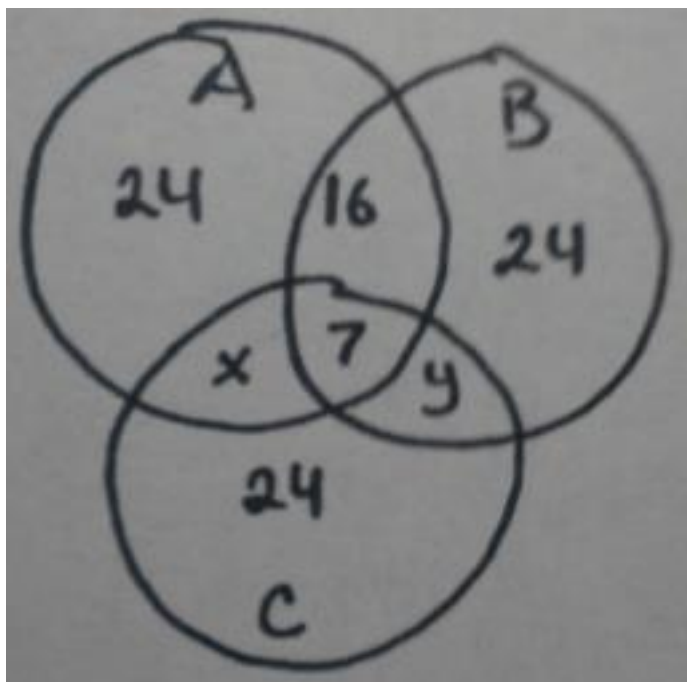
$$|U - ((A \cap B) \cap C)|$$

$$|U| - |((A \cap B) \cap C) \cap U|$$

$$|U| - |((A \cap B) \cap C)|$$

$$50 - 7 = 43$$

c) The following picture is consistent with the given information. Note: the picture is drawn differently than most Venn diagrams. For example, the 24 in A, B and C represent the total number of items in A, B and C and NOT the items only exclusively in A, B and C, respectively. The 16 represents the number of items shared by A and B, INCLUDING items that are also in C. (The typical drawing would put a 9 where the 16 is.) Since 16 elements are shared between A and B, there are 8 elements in A not in B and vice versa. Unfortunately, the given information doesn't restrict these elements. Namely, they might only be in A, OR they could be shared with A and C. In the picture, x represents the number of elements shared between A and C and y represents the number of elements shared between B and C.



Assigning x and y to the missing values, we can see that:

$$\text{Since } |A| = 24, 24 \geq 16 + x - 7$$

$$x \leq 15$$

$$\text{Since } |B| = 24, 24 \geq 16 + y - 7$$

$$y \leq 15$$

Solving IE for  $|((A \cup B) \cup C)|$  for  $|A \cap C| + |B \cap C|$ ,

$$\text{we get that } |A \cap C| + |B \cap C| = 23$$

$$\text{And since } |A \cap B \cap C| = 7, x \geq 7, y \geq 7$$

Valid combinations of x and y are

(15, 8), (14, 9), (13, 10), (12, 11), (11, 12), (10, 13), (9, 14), (8, 15)

**(Grading: 1 pt for not possible, 1 pt each for two possible pictures)**

d) Using the Inclusion-Exclusion Principle, we have: **(Grading: 2 pts for Inc/Exc, 1 answer)**

$$|A \cup B| = |A| + |B| - |(A \cap B)| = 24 + 24 - 16 = 32$$

e) Using the definition of set difference, we derive the following: **(Grading: 2 pts setup, 1 answer)**

$$\begin{aligned} |((C - A) - B)| &= |((A \cup B) \cup C) - (A \cup B)| \\ &= |((A \cup B) \cup C)| - |((A \cup B) \cup C) \cap (A \cup B)| \\ &= 40 - 32 = 8 \end{aligned}$$

3) Here is the equivalence, using Set Laws: **(Grading: 10 pts for both parts, 5 pts for set law proof, take off 1 per error cap at 5)**

$$\begin{aligned} &A \cup ((A \cup C) \cap B \cap \neg(\neg A \cap C)) \\ &A \cup ((A \cup C) \cap B \cap (\neg\neg A \cup \neg C)) \text{DeMorgan's} \\ &A \cup ((A \cup C) \cap B \cap (A \cup \neg C)) \text{Double Negation} \\ &A \cup ((A \cup C) \cap (A \cup \neg C) \cap B) \text{Commutative} \\ &A \cup ((A \cup (C \cap \neg C)) \cap B) \text{Distributive} \\ &A \cup ((A \cup \emptyset) \cap B) \text{Inverse} \\ &A \cup (A \cap B) \text{Identity} \\ &A \text{Absorption} \end{aligned}$$

Here is the equivalence proven by a Set Table:

A	B	C	$A \cup C$	$\neg A \cap C$	$\neg(\neg A \cap C)$	$A \cup ((A \cup C) \cap B \cap \neg(\neg A \cap C))$
0	0	0	0	0	1	0 (due to B = 0)
0	0	1	1	1	0	0 (due to B = 0)
0	1	0	0	0	1	0 (due to A = 0, C = 0)
0	1	1	1	1	0	0 (due to second to last column = 0)
1	0	0	1	0	1	1
1	0	1	1	0	1	1
1	1	0	1	0	1	1
1	1	1	1	0	1	1

The equivalence is proven because the first and last columns are identical under all possible positions of elements with respect to sets A, B and C.

**(Grading: 5 pts total, grade proportionally, giving a whole number of points)**

4) We use direct proof as follows: **(Grading: 5 pts total)**

Let x be an arbitrary element of (B-A) **1 pt**

$$x \in (B - A) \rightarrow x \in B \text{ and } x \in \neg A$$

$$\text{Since } B \subseteq C, x \in C$$

$$\text{So, } x \in C \text{ and } x \in \neg A$$

**1 pt for def of subtraction, 1 pt subset, 1 pt x in C - A**

$$\text{Therefore, } x \in (C - A)$$

Since an arbitrary x from (B-A) must also be an element of (C-A), we can conclude that

$$(B - A) \subseteq (C - A) \quad \text{1 pt conclusion}$$

5) The inverse of the statement is false. Here is a counter-example disproving the given proposition. **(Grading: 5 pts total)**

Let  $A = \{1, 2\}$ ,  $B = \{1\}$ ,  $C = \{2\}$      **2 pts for saying false, 2 pts for listing sets, 1 pt explanation**

$$\{1\} - \{1, 2\} \equiv \emptyset$$

$$\{2\} - \{1, 2\} \equiv \emptyset$$

$$\emptyset \subseteq \emptyset$$

but,  $\{1\} \not\subseteq \{2\}$

6) This statement is false as well. The key is that even though A and B have strictly fewer elements than C, it's possible that their intersection does not. But, the intersection can't contain any elements that C doesn't have. As this counter-example shows, it may be equal, however. **(Grading: 5 pts total)**

Let  $A = \{1\}$ ,  $B = \{2\}$ , and  $C = \{1, 2\}$      **2 pts for saying false, 2 pts for listing sets, 1 pt explanation**

$$\{1\} \subset \{1, 2\}$$

$$\{2\} \subset \{1, 2\}$$

$$\{1\} \cup \{2\} = \{1, 2\} \not\subset \{1, 2\}$$