# <u>Spring 2014 COT 3100 Free Response Exam #2 Solutions</u>

1) (10 pts) Use the Extended Euclidean Algorithm to find $72^{-1}$ (mod 239). Please give your answer as an integer in between 0 and 238, inclusive. A majority of the grade will be given for your work and not the final answer.

<u>Solution</u>
First, run the regular Euclidean Algorithm:

239 = 3 x 72 + 23
72 = 3 x 23 + 3
23 = 7 x 3 + 2
3 = 1 x 2 + 1 **(2 pts total to get here)**
2 = 2 x 1 (Note: This line isn't necessary for a student solution)

Write the second to last equation backwards and start the substitution process using the previous equations written backwards as necessary, collecting like terms after each substitution:

3 – 1 x 2 = 1 **(1 pt)**
3 – (23 – 7 x 3) = 1
3 – 23 + 7 x 3 = 1 **(2 pt)**
8 x 3 – 1 x 23 = 1
8(72 – 3 x 23) – 1 x 23 = 1
8 x 72 – 24 x 23 – 1 x 23 = 1
8 x 72 – 25 x 23 = 1 **(2 pt)**
8 x 72 – 25(239 – 3 x 72) = 1
8 x 72 – 25 x 239 + 75 x 72 = 1
83 x 72 – 25 x 239 = 1 **(2 pt)**

Looking at both sides of this equation mod 239, we find

83 x 72 ≡ 1 (mod 239)

It follows that $72^{-1} \equiv 83$ mod (239). **(1 pt)**

2) (10 pts) Prove or disprove: Let a, b and c be arbitrary positive integers greater than 1. If gcd(a, b) = 1 and gcd(b, c) = 1, then gcd(a, c) = 1.

Solution

The claim is false. Let a = 2, b = 1 and c = 2. In this counter-example we find that gcd(a, b) = gcd(2, 1) = 1, gcd(b, c) = gcd(1, 2) = 1, but gcd(a, c) = gcd(2, 2) = 2.

Thus, relative primality is not transitive. (Namely, just because a common item is relatively prime with two different ones in the order stated above doesn't mean that the first and third are related in the same way, for this particular relationship.)

**Grading: 3 pts for stating the result is false clearly, maximum 2 pts for any proof, 7 pts for giving a counter-example and clearly showing that it is one (3 pts for the values of a, b and c, and 4 pts for plugging into the assertion and showing the if part holds but the then part doesn't for that one example.)**

3) (10 pts) Let $H_n$ denote the $n^{th}$ Harmonic number. Use mathematical induction on n to show that $H_{2^n} \geq 1 + \frac{n}{2}$, for all non-negative integers n.

Solution

Base case: n = 0. LHS = $H_{2^0} = H_1 = 1$. RHS = $1 + \frac{n}{2} = 1 + \frac{0}{2} = 1$. Since the two sides are equal, the given formula is true for n = 0. **(1 pt)**

Inductive hypothesis: Assume for an arbitrarily chosen non-negative integer n = k that $H_{2^k} \geq 1 + \frac{k}{2}$. **(1 pt)**

Inductive step: Prove for n = k+1 that $H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$. **(1 pt)**

$$
\begin{aligned}
H_{2^{k+1}} &= \sum_{i=1}^{2^{k+1}} \frac{1}{i} \\
&= \left(\sum_{i=1}^{2^k} \frac{1}{i}\right) + \left(\sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i}\right) \\
&= H_{2^k} + \left(\sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i}\right) \\
&\geq 1 + \frac{k}{2} + \left(\sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i}\right) \\
&\geq 1 + \frac{k}{2} + \left(\sum_{i=2^k+1}^{2^{k+1}} \frac{1}{2^{k+1}}\right) \\
&= 1 + \frac{k}{2} + \left(\frac{1}{2^{k+1}} \sum_{i=2^k+1}^{2^{k+1}} 1\right) \\
&= 1 + \frac{k}{2} + \left(\frac{2^k}{2^{k+1}}\right) \\
&= 1 + \frac{k}{2} + \left(\frac{1}{2}\right) \\
&= 1 + \frac{k+1}{2}
\end{aligned}
$$

This completes the inductive step. Thus, we can conclude for all non-negative integers n, $H_{2^n} \geq 1 + \frac{n}{2}$.

**Rest of the Grading: 1 pt sum split, 2 pts use IH, 1 pt sum simplify with $\frac{1}{2^{k+1}}$, 1 pt number of terms in second sum, 2 pts rest of algebraic simplification.**

4) (10 pts) Let $f(n) = \frac{n}{n+2}$. Define $f^k(n)$ to be the function f composed with itself k times. More formally, $f^0(n) = n$ and $f^k(n) = f(f^{k-1}(n))$, for all positive integers $k$. Using induction on $k$, prove that for all positive integers $k$, $f^k(n) = \frac{n}{(2^k-1)n+2^k}$. (Hint: The algebra can be messy if you don't multiply both your numerator and denominator by $(2^k - 1)n + 2^k$. So, in full, after you do a particular step, you would take your fraction and multiply it by $\frac{(2^k-1)n+2^k}{(2^k-1)n+2^k}$. Please feel free to ignore the hint, but I do think it reduces the amount of algebra drastically.)

Solution

Base case: k = 1. LHS = $f^1(n) = f\left(f^0(n)\right) = f(n) = \frac{n}{n+2}$

RHS = $\frac{n}{(2^1-1)n+2^1} = \frac{n}{n+2}$

For all values of n, these two functions are equal, so the base case holds. **(1 pt)**

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer k = m that $f^m(n) = \frac{n}{(2^m-1)n+2^m}$. **(1 pt)**

Inductive Step: Prove for k = m+1 that $f^{m+1}(n) = \frac{n}{(2^{m+1}-1)n+2^{m+1}}$ **(1 pt)**

$$f^{m+1}(n) = f(f^m(n))$$

$$= f\left(\frac{n}{(2^m-1)n+2^m}\right), \text{ using I. H.}$$

$$= \frac{\frac{n}{(2^m-1)n+2^m}}{\frac{n}{(2^m-1)n+2^m}+2}$$

$$= \frac{\frac{n}{(2^m-1)n+2^m}}{\frac{n}{(2^m-1)n+2^m}+2} \times \frac{(2^m-1)n+2^m}{(2^m-1)n+2^m}$$

$$= \frac{n}{n+2((2^m-1)n+2^m)}$$

$$= \frac{n}{n+(2^{m+1}-2)n+2^{m+1}}$$

$$= \frac{n}{(2^{m+1}-1)n+2^{m+1}}$$

This completes the inductive step. Thus, we can conclude for all positive integers k, $f^k(n) = \frac{n}{(2^k-1)n+2^k}$.

**Rest of the Grading: 1 pt function break down, 2 pts sub IH, 2 pts plug into sub accurately, 2 pts rest of the algebra**