Sections in the book

1 Foundations: Logic & Proofs

2 Basic Structures

3.4, 3.5 Integers

4.1 – 4.3 Induction

5 Counting

8 Relations (partial ordering)

not necessary The Principle of Well-Ordered Induction + Topological Sorting

Part A

Induction

Sets

Logic

Counting

Relations

Functions

Number Theory ✓

Laws of Logic

Laws of Set Theory

Rules of Inference

Induction Fall 2009 Part A Problem 1

$$\sum_{i=1}^{n} (3i+1) = \frac{n(3n+5)}{2} \text{ for all } n \geq 1.$$

Base case: $n=1$ $\quad \underbrace{(3+1)}_{4} = \underbrace{\frac{3+5}{2}}_{4}$

Inductive hypothesis: assume for an arbitrary positive integer $n=k$ that

$$\sum_{i=1}^{k} (3i+1) = \frac{k(3k+5)}{2}.$$

Inductive step: prove that

$$\sum_{i=1}^{k+1} (3i+1) = \frac{(k+1)(3(k+1)+5)}{2} \quad \text{holds for } n=k+1.$$

$$\sum_{i=1}^{k+1} (3i+1) = \sum_{i=1}^{k} (3i+1) + [3(k+1)+1]$$

$$= \frac{k(3k+5)}{2} + [3(k+1)+1] \quad \text{because of the ind. hypothesis}$$

$$= \frac{k(3k+5)}{2} + 3k+4$$

$$= \frac{k(3k+5)+6k+8}{2}$$

$$= \frac{3k^2 + 5k + 6k + 8}{2}$$

$$= \frac{3k^2 + 11k + 8}{2}$$

$$= \frac{(k+1)(3k+8)}{2}$$

$$= \frac{(k+1)(3k+8)}{2}$$

$$= \frac{(k+1)(3(k+1)+5)}{2}$$

$k = 1$ $\qquad$ $LHS(1) = RHS(1)$

base case

Ind. hypothesis +

Ind. step

$$\forall k \quad LHS(k) = RHS(k) \longrightarrow LHS(k+1) = RHS(k+1)$$

$$\sum_{k=1}^{n}(3k+1) = \left[3\sum_{k=1}^{n}k\right] + n$$

$$3\,\frac{n(n+1)}{2} \cdot n$$

Summer 2009 Part A Problem 1

Using proof by induction, prove that

$8 \mid (3^{2n+1} + 5^{2n+1})$ for all

non-negative integers.

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$8 \mid m$ equivalent to

$m \equiv 0 \pmod 8$

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inequalities $\sum \dots \leq \dots$

$$3^{2(k+1)+1} + 5^{2(k+1)+1}$$

$$3^2\left(3^{2k+1}\right) + 5^2\left(5^{2k+1}\right)$$

$$-\left(3^{2k+1}\right) - \left(5^{2k+1}\right)$$

$$8\left(3^{2k+1}\right) + 24\left(5^{2k+1}\right)$$

$$a + b \equiv 0 \mod 8$$

$$\not\Longrightarrow \quad ma + nb \equiv 0$$

Induction hypothesis: Assume that

$8 \mid \boxed{3^{2k+1}} + \boxed{5^{2k+1}}$ holds.

Inductive step:

We have to show that

$8 \mid 3^{2(k+1)+1} + 5^{2(k+1)+1}$ holds

$$\circledast \quad 3^{2(k+1)+1} + 5^{2(k+1)+1}$$

$$= 3^2 \boxed{3^{2k+1}} + 5^2 \boxed{5^{2k+1}}$$

we have to factor out $3^2$ and $5^2$ so that $\boxed{\phantom{x}}$ match with Ind. hypothesis.

$$= 9 \cdot 3^{2k+1} + 25 \cdot 5^{2k+1}$$

$25 = 9 + 16$

$$= 9 \cdot 3^{2k+1} + 9 \cdot 5^{2k+1} + 16 \cdot 5^{2k+1}$$

$$= \underbrace{9\left(3^{2k+1} + 5^{2k+1}\right)}_{\text{divisible by 8 because of ind. hypothesis}} + \underbrace{16}_{\text{divisible by 8}} \cdot 5^{2k+1}$$

therefore $*$ divisible by 8 since it is the sum of mult. of 8