Foundation Exam

Part A

- Induction
- Sets
- Logic

$$\begin{array}{l} P_1 \\ P_2 \\ \vdots \\ P_k \\ \hline \therefore C \end{array}$$

Part B

- Counting
- Relations

  reflexive, symmetric, transitive, antisym
- Functions

  injective, surjective, bijective, compose $g \circ f$
- Number theory

  gcd, lcm, prime factors, mod, div, divides |

  gcd Euclid alg.

Relations

reflexive
symmetric
antisymmetric
transitive
equivalence relation
order relation

$\leqslant$ reflexive
anti-symmetric
transitive

$S \circ R$

proof techniques

direct
indirect
contradiction

NT

Fall 2009 Part B Problem 7

Prove that for every positive integer x of exactly four digits, if the sum of the digits is divisible by 3, then x itself is divisible by 3.

$x = 6132$

sum of the digits is

$6 + 1 + 3 + 2 = 12$

$$X = 10^3 x_3 + 10^2 x_2 + 10^1 x_1 + x_0$$

6132 $\quad x_3 = 6 \quad x_2 = 1 \quad x_1 = 3 \quad x_0 = 2$

assume that $3 \mid x_3 + x_2 + x_1 + x_0$

$$\Rightarrow x_3 + x_2 + x_1 + x_0 \equiv 0 \pmod{3}$$

$$\overset{*}{\Rightarrow} x_3 + 999 x_3 + x_2 + 99 x_2 + x_1 + 9 x_1 + x_0 \equiv 0 \pmod{3}$$

$$\Rightarrow 1000 x_3 + 100 x_2 + 10 x_1 + x_0 \equiv 0 \pmod{3}$$

$$\Rightarrow 3 \mid X$$

* is valid because we add a multiple of 3 and so do not change the remainder when dividing by 3

Euclid's algorithm

$$\gcd(a, b) = \gcd(b, a \bmod b)$$

Summer 2009 Part B 7

(a) Write 11! in its prime factorized form.

$$11! = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11$$

$$= 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot 2 \cdot 3 \cdot 7 \cdot 2^3 \cdot 3^2 \cdot 2 \cdot 5 \cdot 11$$

$$= 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$$

$(3!)!$

(b) Find all integers n greater than 3 for which $n^2 - 9$ is a prime number. Prove that no other answers exist.

$n = 4$ $\qquad n^2 - 9 = 4^2 - 9 = 16 - 9 = 7$

$n = 5$ $\qquad 5^2 - 9 = 25 - 9 = 16$ not prime

$$(n^2 - 9) = \underbrace{(n-3)}_{x}\underbrace{(n+3)}_{y}$$

For $n \geqslant 5$, we can write $n^2 - 9$ as $x \cdot y$ with $x, y > 1$. There $n^2 - 9$ cannot be a prime.

Fall 2009 A 2

$$(p \wedge r) \vee (q \wedge r)$$
$$= (p \vee q) \wedge r$$

A, B, C arbitrary

$$(B-A) \cup (C-A) = (B \cup C) - A$$

$= \{x \mid x \in (B-A) \vee (C-A)\}$ def. of union

$= \{x \mid (x \overset{p}{\in} B \wedge x \overset{r}{\notin} A) \vee (x \overset{q}{\in} C \wedge x \overset{r}{\notin} A)\}$ def. of difference

$= \{x \mid (x \overset{p}{\in} B \vee x \overset{q}{\in} C) \wedge x \overset{r}{\notin} A\}$ distr. law

$= \{x \mid x \in B \cup C \wedge x \notin A\}$ def. of union

$= \cancel{\{x \mid x}$ $(B \cup C) - A$ def. of difference