# Lecture 22

A relation on a set $A$ is called an <u>equivalence relation</u> if it is

(i) reflexive

(ii) symmetric

(iii) transitive

Congruence relation

$m$ is a positive integer

$R \subseteq \mathbb{Z} \times \mathbb{Z}$

$R = \{ (a, b) \mid a \equiv b \pmod{m} \}$

$a \equiv b \pmod{m}$ means that $m \mid (a-b)$

$\exists k \quad a = b + km$

(i) reflexive $\forall a\, aRa$

$aRa$ is true for all integers $a$ since $m \mid (a-a)$

(ii) symmetric $\forall a \forall b\ aRb \to bRa$

Assume $aRb$.

Def. of $R$

$a \equiv b \pmod{m}$

Def. of $\equiv$

$a = b + km$ for some integer $k$

$b = a - km = a + (-k)m$

$b \equiv a$

$bRa$

(iii) transitive $\forall a \forall b \forall c \; aRb \wedge bRc \rightarrow aRc$

assume $aRb$ and $bRc$

$a \equiv b \wedge b \equiv c$

integers

$a = b + km \wedge b = c + lm$ for some $k, l$

$a = c + lm + km = c + (k+l)m$

$a \equiv c$

$aRc$

$R = \{ (a,b) \mid a^2 + b \text{ even} \}$

(i) reflexive

$aRa$ is true iff $a^2 + a$ is even.

- assume $a$ is even $\quad a = 2k$ for some $k$

$$\begin{aligned} a^2 + a &= 4k^2 + 2k \\ &= 2(2k^2 + k) \\ &= 2l \qquad l = 2k^2 + k \end{aligned}$$

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- assume $a$ is odd

$a = 2k+1$ for some $k$

$$\begin{aligned} a^2 + a &= 4k^2 + 4k + 1 + 2k + 1 \\ &= 2(2k^2 + 3k + 1) \\ &= 2\ell \qquad \ell = 2k^2 + 3k + 1 \in \mathbb{Z} \end{aligned}$$

$a^2 + a$ is always even

$a^2$ even p $\Longrightarrow$ $a$ even q

$a^2$ even $\quad a^2 = 2k$ for some integer $k$

$$a = \sqrt{2k}$$

direct proof

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contraposition $\quad p \to q$ is equivalent to

$$\neg q \to \neg p$$

$R \subseteq \mathbb{R} \times \mathbb{R}$

$xRy$ iff $|x-y| \leq 1$

(i) $|x-x| = |0| = 0 \leq 1$ reflexive

(ii) $|x-y| \leq 1 \longrightarrow |y-x| \leq 1$ symmetric

transitive

$|0-1| \leq 1 \wedge |1-2| \leq 1$ but $|0-2| \not\leq 1$

$0R1$ $1R2$ $0 \not R 2$

# Equivalence classes

$R$ an equivalence relation on $A$

$a \in A$

$[a]_R$ the equivalence class of $a$

$[a]_R = \{ b \mid aRb \}$

$$a \equiv b \pmod{7}$$

$$[0] = \{0, \pm 7, \pm 14, \pm 21, \ldots\}$$

$$[7] = [0]$$

$$[1] = \{1, 8, 15, 22, \ldots\} \cup$$
$$\{-6, -13, -20, \ldots\}$$

$$[-6] = [1]$$

$$a \equiv b \pmod{m}$$

$$[k] = [\ell] \quad \text{if} \quad k \equiv \ell \pmod{m}$$

## Theorem 1

$R$ is an equivalence relation on $A$

These statements for elements $a$ and $b$ are equivalent

- $aRb$
- $[a] = [b]$
- $[a] \cap [b] \neq \emptyset$

$A$ set

a partition of $A$ is a collection of <u>subsets</u> $A_1, A_2, A_3, \ldots$ of $A$

- $A_i \neq \emptyset$
- $A_i \cap A_j = \emptyset$ when $i \neq j$
- $\bigcup_{i \in I} A_i = A$

$$A = \{1, 2, 3, 4, 5\}$$

$$A_1 = \{1, 2, 3\}$$

$$A_2 = \{4\}$$

$$A_3 = \{5\}$$

Theorem 2:

Let $R$ be an equivalence relation on $A$. Then the equivalence classes of $R$ form a partition of $A$.

Ex: $aRb$ iff $a \equiv b \mod 3$

$[0]$ $[1]$ $[2]$

Conversely, given a partition $\{A_i \mid i \in I\}$ of $A$, there is an equivalence relation $R$ that has the sets $A_i$, $i \in I$, as its equivalence classes.

$A = \{1, 2, 3, 4, 5\}$

$A_1 = \{1, 2, 3\}$

$A_2 = \{4\}$

$A_3 = \{5\}$

$R \subseteq A \times A$

$$R = \{ \overbrace{(1,1), (2,2), (3,3)}^{\text{reflexive}}, (1,2), (2,1), (1,3), (3,1), (2,3), (3,2) \}$$

$$\cup \{(4,4)\} \cup \{(5,5)\}$$