Lecture 20

Relations

Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

R the relation $R \subseteq A \times B$

- aRb to denote $(a,b) \in R$
  a is related to b

- $a \not R b$ $(a,b) \notin R$
  a is not related to b

Students $\qquad$ Courses

$S = \{DANIEL, \ldots\}$ $\quad C = \{COT3100, COT4810, \ldots\}$

E enrolled

$(\text{Daniel}, COT3100) \in E$

$(\text{Pawel}, COT3100) \notin E$

Functions as Relations

$$f : A \longrightarrow B$$

the corresponding relation $R_f \subseteq A \times B$

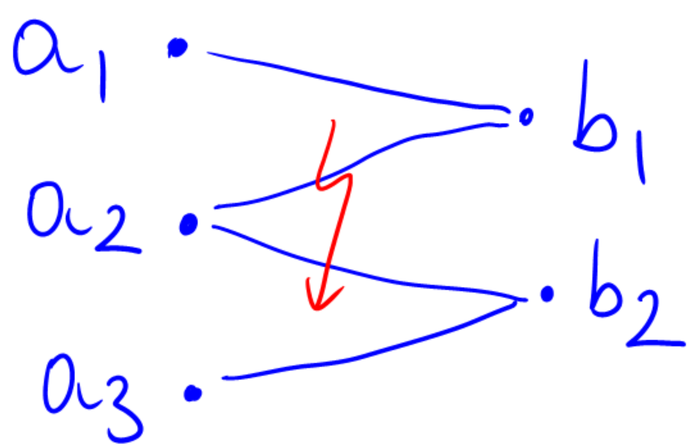
$$\left.\begin{array}{l} a R_f b \\ (a,b) \in R_f \end{array}\right\} \text{iff } f(a) = b$$

An example of a relation that does not correspond to any function.

$$A = \{a_1, a_2, a_3\} \qquad B = \{b_1, b_2\}$$

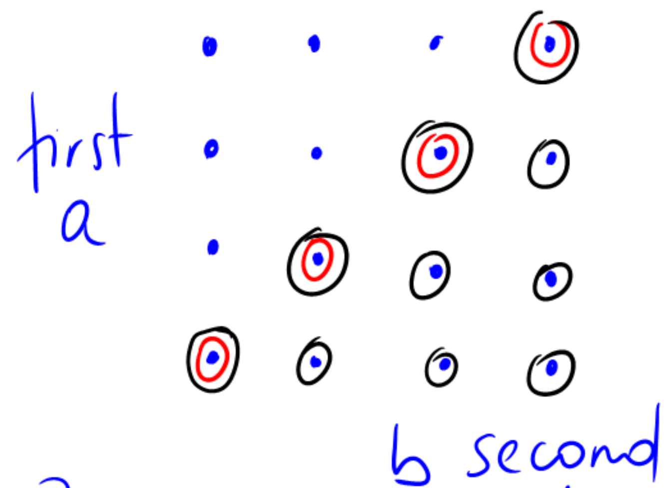
$$R = \{(a_1, b_1), (a_2, b_1), (a_2, b_2), (a_3, b_2)\}$$

domain codomain



Relations on a Set

$R \subseteq A \times A$



$A = \mathbb{N}$

$R_1 = \{(a,b) \mid a \leq b\} \subseteq \mathbb{N} \times \mathbb{N}$

$a \leq b$

$R_2 = \{(a,b) \mid a = b\}$

Properties of Relations

$R \subseteq A \times A$ is called <u>reflexive</u> iff

$\forall a \ (a,a) \in R$

- $R_=$ (R_{equal}) $= \{(a,b) \mid a=b\}$ reflexive symmetric
- $R_\geq = \{(a,b) \mid a \geq b\}$ reflexive not symmetric
- $R_> = \{(a,b) \mid a > b\}$ irreflexive not symmetric
- $R_{divides} = \{(a,b) \mid a \mid b\}$ reflexive not symmetric

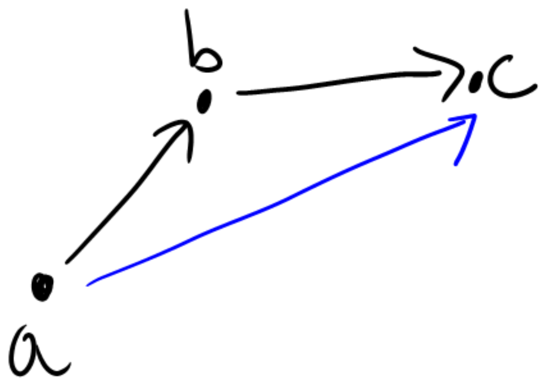
$R \subseteq A \times A$ is called symmetric iff
$\forall a \forall b \; (a,b) \in R \rightarrow (b,a) \in R$

$R \subseteq A \times B$ is called antisymmetric iff
$\forall a \forall b \; (a,b) \in R \wedge (b,a) \in R \rightarrow a = b$

- $\leqslant$ is antisymmetric
  $a \leqslant b \wedge b \leqslant a \longrightarrow a = b$
- $|$ (divides) is antisymmetric
  $a \mid b \wedge b \mid a \longrightarrow a = b$

$R \subseteq A \times A$ is called transitive iff

$$\forall a \forall b \forall c \quad (a,b) \in R \land (b,c) \in R \rightarrow \underline{(a,c) \in R}$$



Combining Relations

Because relations from A to B are subsets of $A \times B$, two relations can be combined in any way two sets can be combined.

$R_1 \cap R_2$

$R_1 \cup R_2$

$R_1 - R_2$

$$R_{\leq} = \{(a,b) \mid a \leq b\} \subseteq \mathbb{N} \times \mathbb{N}$$

$$R_{\geq} = \{(a,b) \mid a \geq b\}$$

$$R_{=} = \{(a,b) \mid a = b\}$$

$$R_{\leq} \cap R_{\geq} = R_{=}$$

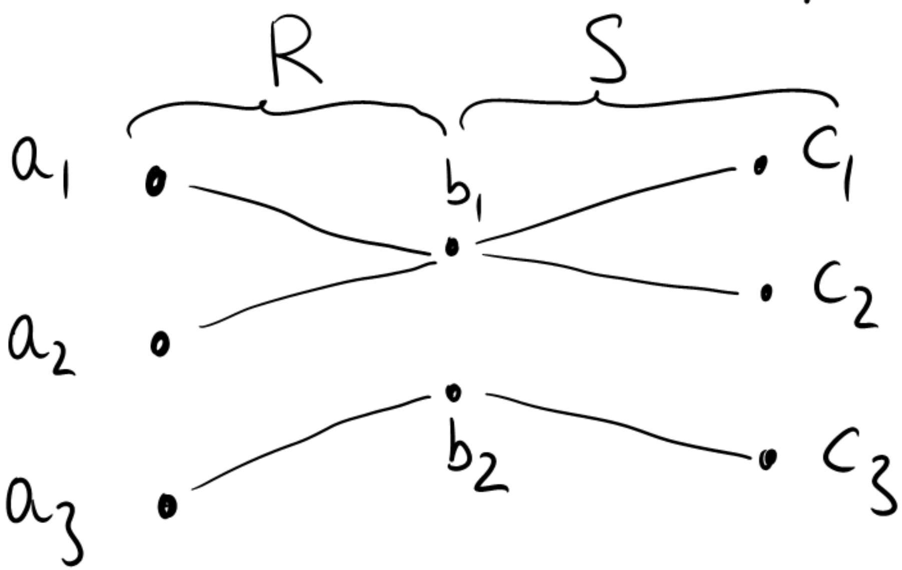
$$R_{\leq} \cup R_{\geq} = \mathbb{N} \times \mathbb{N}$$

$$R_{\leq} - R_{=} = R_{<}$$

Composition of relations

$R \subseteq A \times B$, $S \subseteq B \times C$, $S \circ R \subseteq A \times C$

$$\boxed{S \circ R} = \{ (a,c) \mid \exists b \in B \ (a,b) \in R \wedge (b,c) \in S \}$$



Lecture 21

HW 20%

1st EXAM 25%

2nd EXAM 25%

FINAL EXAM 30%

$$0.2 \times \left(\frac{\text{score you got}}{\text{total score for HW}} \right) + 0.25 \times \left(\frac{}{100} \right) +$$

$$0.25 \times \left(\frac{}{100} \right) + 0.3 \left(\frac{}{120} \right)$$

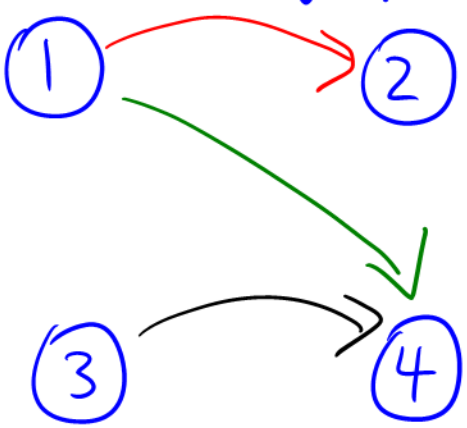
Representing Relations

$A = \{1,2,3,4\}$

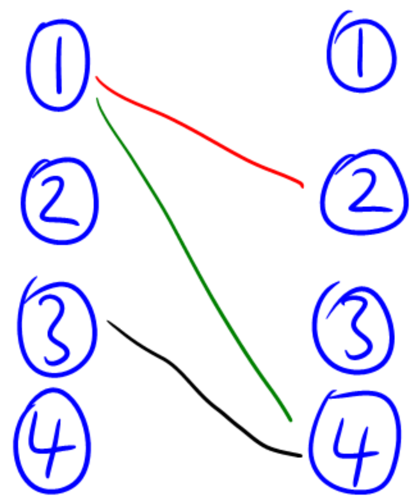
$R \subseteq A \times A$

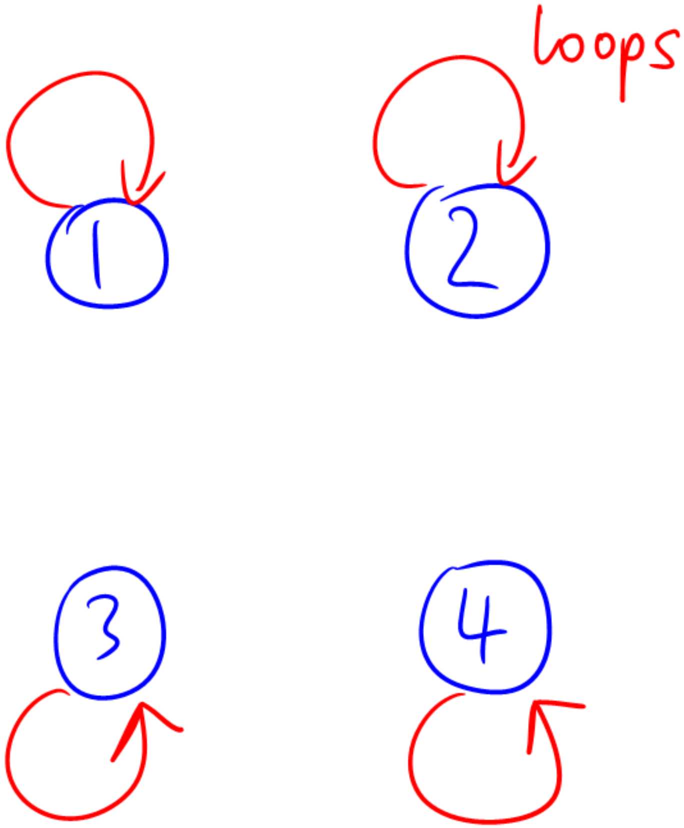
$$R = \{\boxed{(1,2)}, \boxed{(3,4)}, \boxed{(1,4)}\}$$

directed graph



$A \times A$





$R \subseteq A \times A$

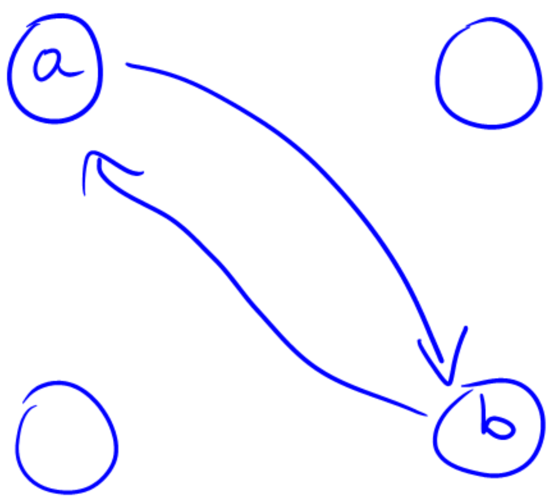
reflexive

$R \subseteq A \times A$

symmetric relation

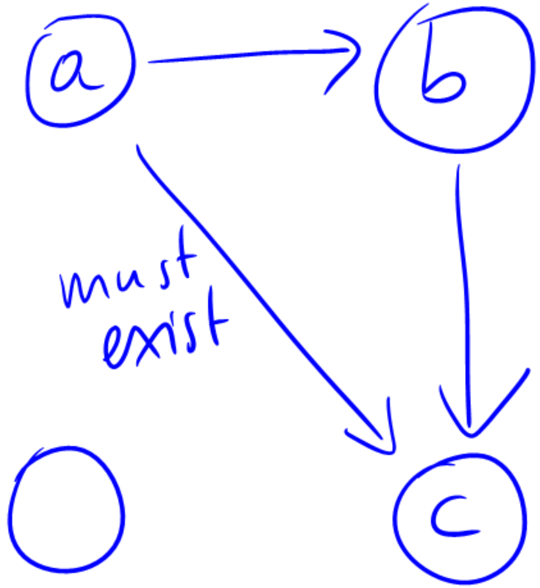
$\forall a \, \forall b \quad (a,b) \in R \longrightarrow (b,a) \in R$

$aRb \qquad\qquad bRa$



$R \subseteq A \times A$

transitive



$R \subseteq A \times A$

antisymmetric

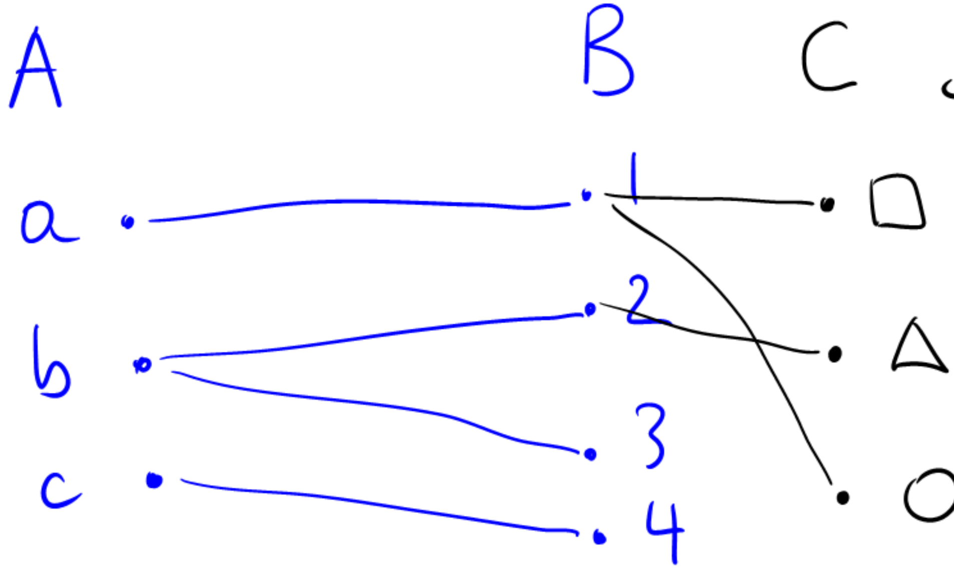
$$\forall a \forall b \ (a,b) \in R \wedge (b,a) \in R \longrightarrow a = b$$

$R \subseteq A \times B$ $S \subseteq B \times C$

$C = \{\square, \triangle, \circ\}$

$A = \{a, b, c\}$ $B = \{1, 2, 3, 4\}$

$R = \{(a,1), (b,2), (c,4), (b,3)\}$



$S = \{(1, \square), (2, \triangle), (1, \circ)\}$

$S \circ R = \{(a, \square), (a, \circ), (b, \triangle)\}$

Closures of Relations

$R \subseteq A \times A$

(reflexive closure of R) $= R \cup \Delta$

$\Delta = \{(a,a) \mid a \in A\}$

symmetric closure

$R \subseteq A \times A$

$R^{-1} = \{(b,a) \mid (a,b) \in R\}$

$(\text{symmetric closure of } R) =$

$R \cup R^{-1}$

Transitive Closure

$S \circ R$

$R \subseteq A \times B \quad S \subseteq B \times C$

$R \subseteq A \times A$

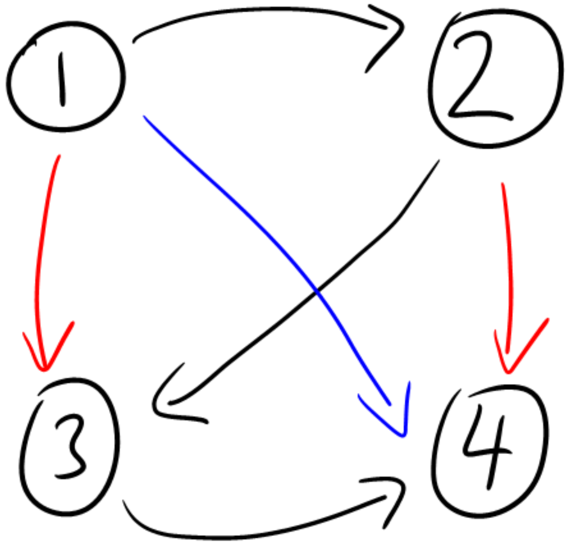
$R^2 = R \circ R \subseteq A \times A \qquad R^n = R^{n-1} \circ R \subseteq A \times A$

$A \times A \quad A \times A$

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

R

$R^2 = R \circ R$



$aR^n b$ iff there is a path of length n from a to b in the graph defined by R

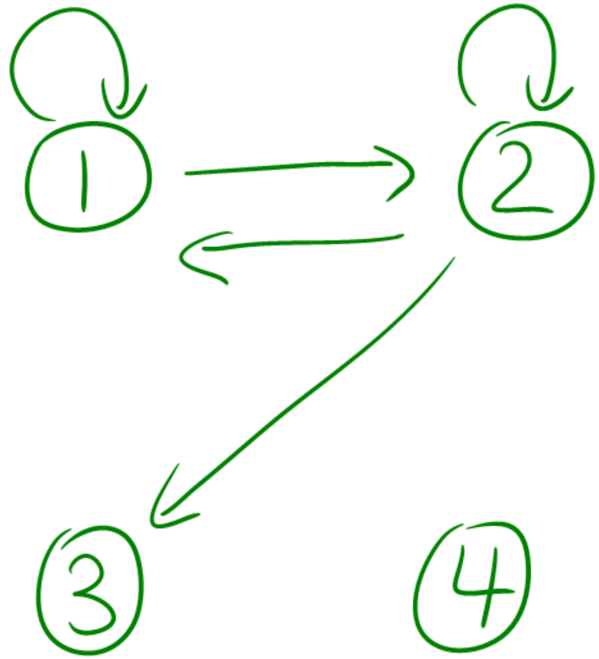
R^3

$R^4 = \emptyset$

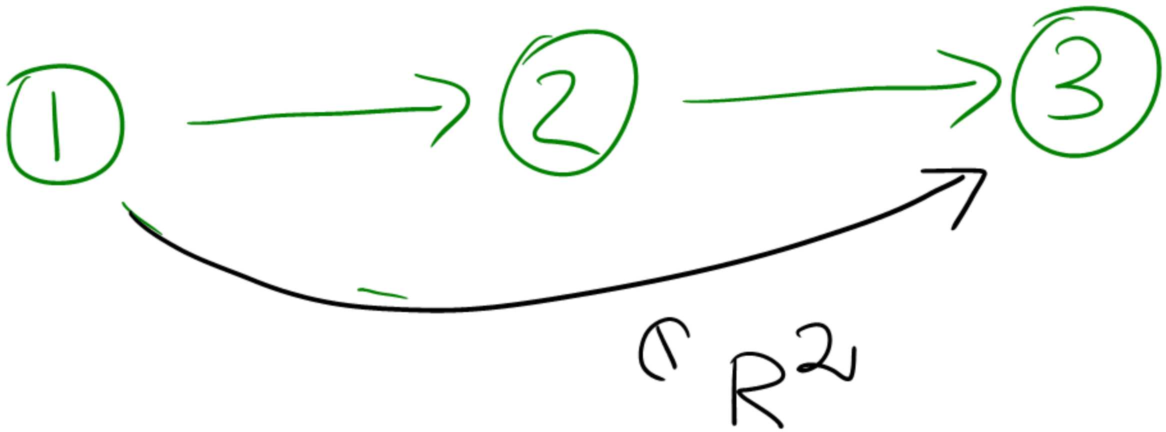
$$R^3 = \{ (a,d) \mid \exists b,c \quad aRb \wedge bRc \wedge cRd \}$$

$$R^3 = R^2 \circ R$$

$$R^2 = R \circ R$$



$(1,3) \in R^7$


1
2
3

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

$$\underline{R} = \{(a,b),(b,c),(c,d)\}$$