Relations

let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

R the relation $R \subseteq A \times B$

- aRb to denote $(a,b) \in R$

  a is related to b

- $a \not R b$ $(a,b) \notin R$

  a is not related to b

Students $\qquad$ Courses

$S = \{\text{DANIEL}, \ldots\}$ $\quad$ $C = \{\text{COT3100}, \text{COT4810}, \ldots\}$

E enrolled

$(\text{Daniel}, \text{COT3100}) \in E$

$(\text{Pawel}, \text{COT3100}) \notin E$

Functions as Relations

$$f : A \to B$$

the corresponding relation $R_f \subseteq A \times B$

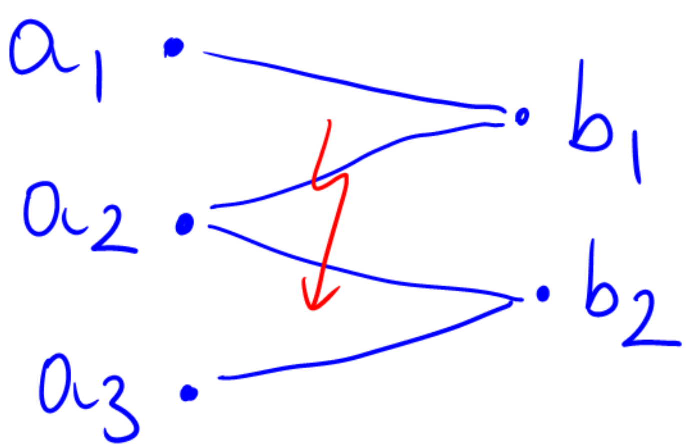
$$\left.\begin{array}{l} a \, R_f \, b \\ (a,b) \in R_f \end{array}\right\} \text{ iff } f(a) = b$$

An example of a relation that does not correspond to any function.

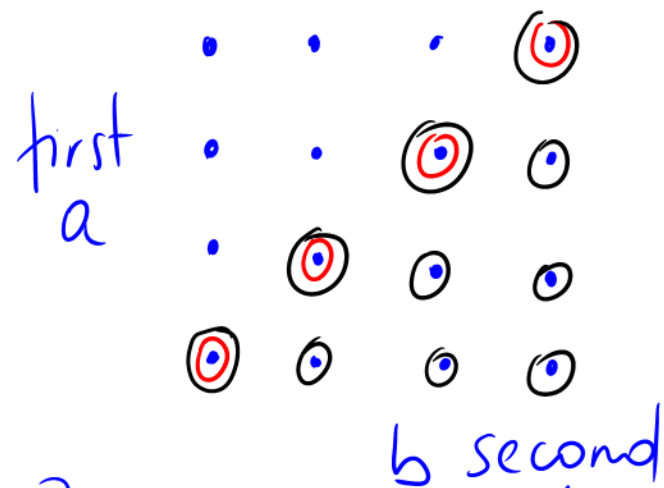
$A = \{a_1, a_2, a_3\}$ $B = \{b_1, b_2\}$

$R = \{(a_1, b_1), (a_2, b_1), (a_2, b_2), (a_3, b_2)\}$

domain codomain



Relations on a Set



$$R \subseteq A \times A$$

$$A = \mathbb{N}$$

$$R_1 = \{(a,b) \mid a \le b\} \subseteq \mathbb{N} \times \mathbb{N}$$

$$a \le b$$

$$R_2 = \{(a,b) \mid a = b\}$$

Properties of Relations

$R \subseteq A \times A$ is called <u>reflexive</u> iff

$$\forall a \ (a,a) \in R$$

- $R_{=} = \{ (a,b) \mid a = b \}$ (Requal) reflexive symmetric
- $R_{\geq} = \{ (a,b) \mid a \geq b \}$ reflexive not symmetric
- $R_{>} = \{ (a,b) \mid a > b \}$ irreflexive not symmetric
- $R_{\text{divides}} = \{ (a,b) \mid a \mid b \}$ reflexive not symmetric

$R \subseteq A \times A$ is called <u>symmetric</u> iff

$\forall a \forall b \ (a,b) \in R \rightarrow (b,a) \in R$

$R \subseteq A \times B$ is called <u>antisymmetric</u> iff

$\forall a \forall b \ (a,b) \in R \wedge (b,a) \in R \rightarrow a = b$

- $\leqslant$ is antisymmetric

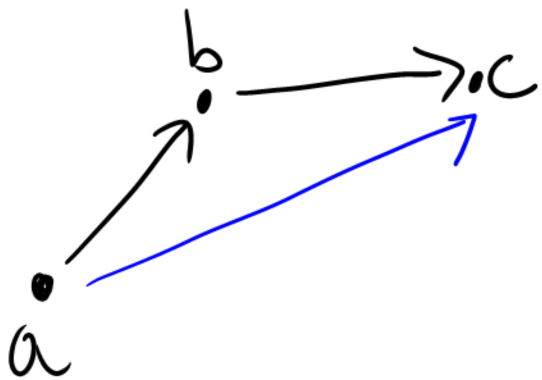
  $a \leqslant b \ \wedge \ b \leqslant a \ \longrightarrow \ a = b$

- $|$ (divides) is antisymmetric

  $a \mid b \ \wedge \ b \mid a \ \longrightarrow \ a = b$

$R \subseteq A \times A$ is called transitive iff

$\forall a \forall b \forall c \quad (a,b) \in R \land (b,c) \in R \rightarrow \underline{(a,c) \in R}$



Combining Relations

Because relations from A to B are subsets of $A \times B$, two relations can be combined in any way two sets can be combined.

$R_1 \cap R_2$

$R_1 \cup R_2$

$R_1 - R_2$

$$R_{\leq} = \{ (a,b) \mid a \leq b \} \subseteq \mathbb{N} \times \mathbb{N}$$

$$R_{\geq} = \{ (a,b) \mid a \geq b \}$$

$$R_{=} = \{ (a,b) \mid a = b \}$$

$$R_{\leq} \cap R_{\geq} = R_{=}$$

$$R_{\leq} \cup R_{\geq} = \mathbb{N} \times \mathbb{N}$$

$$R_{\leq} - R_{=} = R_{<}$$

Composition of relations

$R \subseteq A \times B$, $S \subseteq B \times C$, $S \circ R \subseteq A \times C$

$$\boxed{S \circ R} = \{ (a,c) \mid \exists b \in B \ (a,b) \in R \land (b,c) \in S \}$$