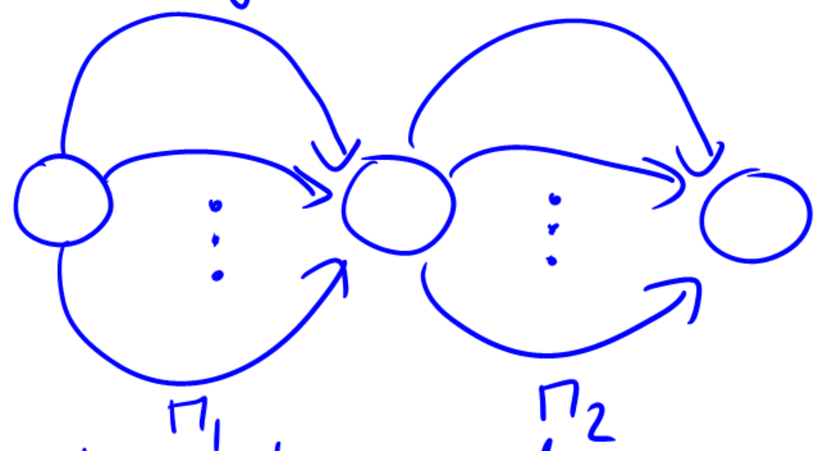
Chapter 5 Counting

Basic Counting Principles



Product rule and sum rule

The Product Rule

Procedure can be broken down into a sequence of two tasks.

Assume there are $\begin{cases} n_1 \\ n_2 \end{cases}$ ways to do the $\begin{cases} \text{first} \\ \text{second} \end{cases}$ task.

Then there are $n_1 \cdot n_2$ ways to do the procedure.

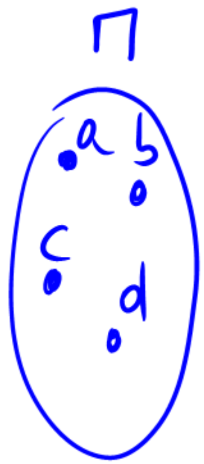
Counting Functions

$|A| = m \quad |B| = n$

How many function $f: A \to B$ are there?

$= m \cdot n$

$= 1$



n^m

$f(1) \quad f(2) \quad f(3)$

a b c d

Counting One-to-One Functions

injective

$f: A \to B$

$|A| = m \qquad |B| = n$

- $n < m \qquad 0$
- $n \geq m$

$$n(n-1)\cdots(n-m+1)$$

$$= \frac{n!}{(n-m)!} = \frac{n(n-1)\cdots(n-m+1)\cancel{(n-m)\cdots 1}}{\cancel{(n-m)\cdots 1}}$$

Counting Subsets of a Finite Set

$$|A| = n$$

$$|P(A)| = 2^n$$

$$A = \{1, 2, 3, \ldots, n\}$$

$$B \subseteq A$$

	1	2	3	4
B	0	0	1	0
B	1	0	1	1

$$n = 4$$

$$B = \{3\}$$

$$B = \{1, 3, 4\}$$

$$|A| = m \quad |B| = n$$

$$|A \times B| = m \cdot n$$

$$|P(A \times B)| = 2^{m \cdot n} \neq 2^{n+m}$$

The Sum Rule

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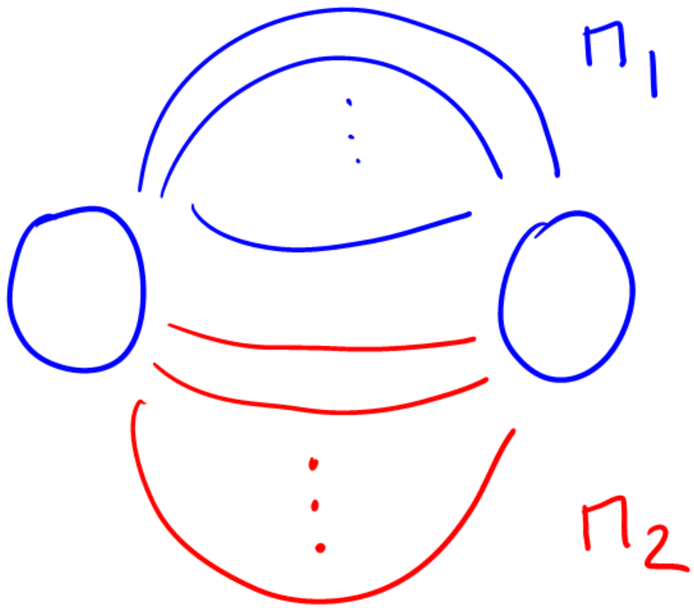


$n_1 + n_2$

A_1 and A_2 are disjoint

$A_1 \cap A_2 = \emptyset$

$|A_1 \cup A_2| = n_1 + n_2$

$|A_1| = n_1$

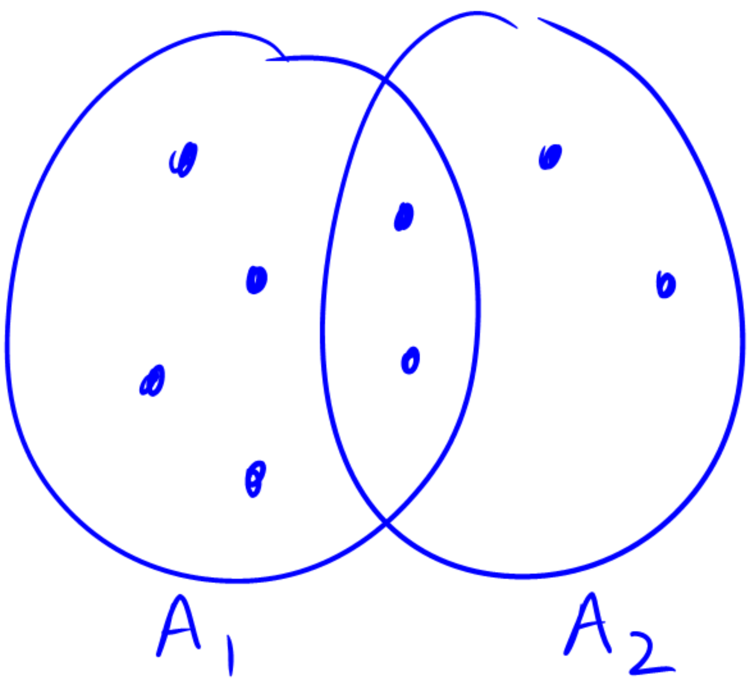
$|A_2| = n_2$

The Inclusion-Exclusion Principle

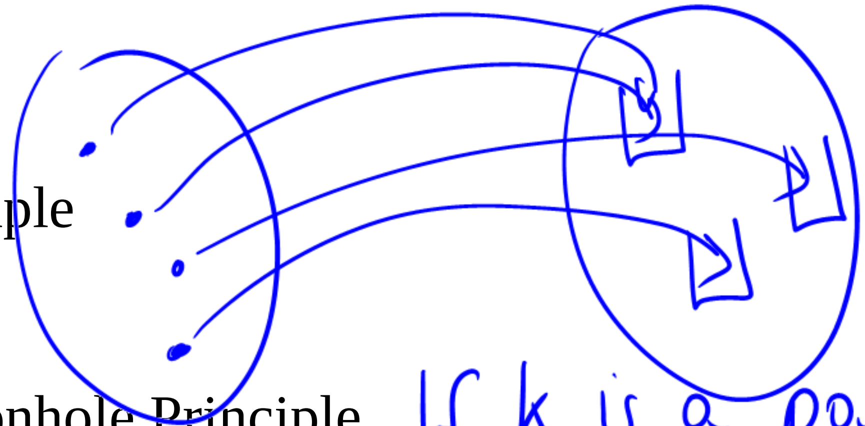
$A_1 \qquad A_2$

$$|A_1 \cup A_2| \leq |A_1| + |A_2|$$

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$



The Pigeonhole Principle



Theorem 1: The Pigeonhole Principle If k is a positive integer and $(k+1)$ or more objects are placed into k boxes, then there is at least one box containing two or more elements.

$|A| \geq k+1$ $\qquad$ $B = |k|$

There cannot exist an injective function $f: A \to B$.

$N \geq k+1$ objects

k boxes

$$\left\lceil \frac{N}{k} \right\rceil \geq \left\lceil \frac{k+1}{k} \right\rceil = 2$$

The Generalized Pigeonhole Principle

Theorem 2: The Generalized Pigeonhole Principle

If N objects are placed into k boxes, then there is at least one box containing at least $\left\lceil \frac{N}{k} \right\rceil$ objects.

Proof by contradiction:

Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects. Then, the total number of objects

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = k\frac{N}{k} = N$$

$5 \cdot 4 \cdot 3 = 60$

Permutations and Combinations

Definition (Permutation):
A permutation of a set of distinct objects is an <u>ordered</u> arrangement of these objects.

An ordered arrangement of r elements is called an <u>r-permutation</u>.

The number of r permutations of a set of n elements is denoted by $P(n,r)$.

Theorem 1: $P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$

$0 \le r \le n$

$$= \frac{n!}{(n-r)!}$$

r

Example 7: How many 8-permutations of the letters contain the string ABC?

8 7

6 5

$\boxed{ABC}DEFGH$

1 2 3 4 5 6

$6! = P(6,6)$

$P(6, r-2)$

Definition (Combination)

An r-combination of elements of a set is an <u>unordered</u> selection of r elements of the set.

Theorem 2:

$$C(n,r) = P(n,r) / r!$$

$$= \frac{n!}{(n-r)!\, r!}$$

$$C(n,r) = C(n, n-r)$$

~~Corollary 2:~~

Relation between combinations and set theory?

$|A| = n$

$A = \{1, 2, 3, 4\}$

How many different subsets $B \subseteq A$ with $|B| = r$ are there?

$$C(n, r)$$

~~Definition 1 (Combinatorial Proof).~~

$$|A| = n$$

$$\sum_{r=0}^{n} C(n,r) = 2^n = |\mathcal{P}(A)|$$

$$\frac{n!}{(n-r)!\,r!}$$

$\{1,2,3\}$ $\qquad C(3,3) = 1$

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$ $\qquad C(3,2) = 3$

$\{1\}$ $\{2\}$ $\{3\}$ $\qquad C(3,1) = 3$

$\emptyset$ $\qquad C(3,0) = 1$

Binomial Coefficients $C(n,r) = \frac{n!}{r!\,(n-r)!} = \binom{n}{r}$

Theorem 1 (The Binomial Theorem):

Let x and y be variables, and n be a non-negative integer.

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}$$

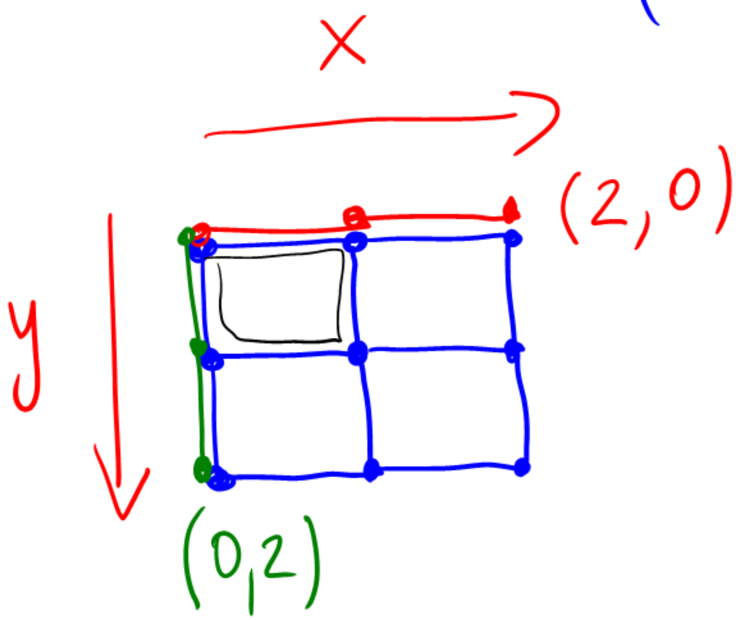
$$= \binom{n}{0} x^n + \binom{n}{1} x^1 y^{n-1} + \cdots + \binom{n}{n} y^n$$

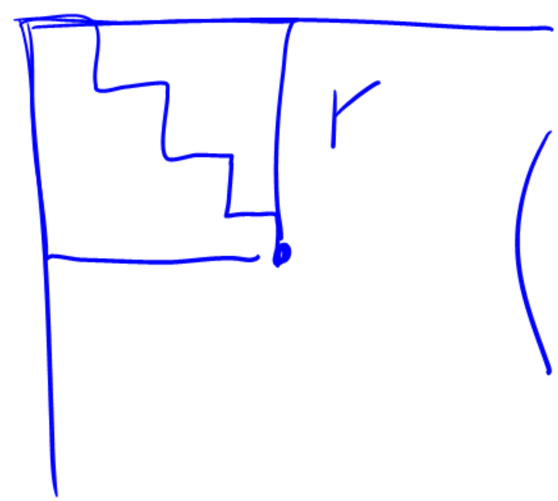
$$(x+y)^2 = 1\cdot x^2 + 2\cdot xy + 1\cdot y^2$$

$$\binom{2}{0} \qquad \binom{2}{1} \qquad \binom{2}{2}$$

$$= (x+y)(x+y) = \boxed{x^2} + xy + yx + \boxed{y^2}$$

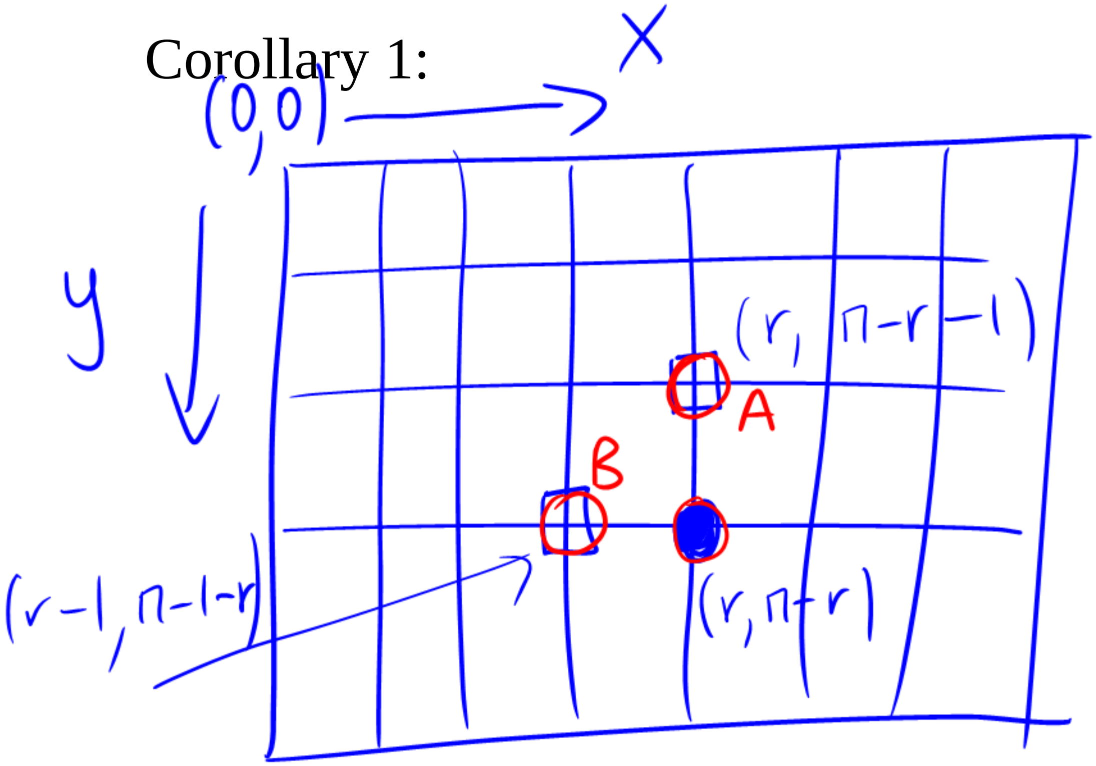
$$= x^2 + 2xy + y^2$$





$$\binom{n}{r}$$

Corollary 1:



$(r, n-r)$

total fare n

$\#$ different paths from $(0,0) \longrightarrow (r, n-r)$ only $\downarrow$ or $\rightarrow$

$$= C(n,r) = \binom{n}{r} = \overbrace{\binom{n-1}{r}}^{A} + \binom{n-1}{r-1}$$

Corollary 2:

Corollary 3:

Pascal’s Identity and Triangle

~~Vandermonde's identity~~

$$C(n, m) = \frac{n!}{m!\,(n-m)!}$$

$$m = r \qquad \frac{n!}{r!\,(n-r)!}$$

$$m = n - r \qquad \frac{n!}{(n-r)!\,(n-(n-r))!}$$

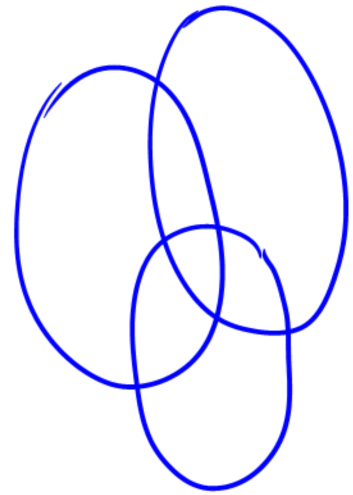
$$= \frac{n!}{(n-r)!\,r!}$$

Generalized Permutations and Combinations

Permutations with Repetion

~~Combinations with Repetition~~

$$|A_1 \cup \underbrace{(A_2 \cup A_3)}_{B}| =$$



$$= |A_1| + |B| - |A_1 \cap B|$$

$$= |A_1| + |A_2 \cup A_3| - |A_1 \cap (A_2 \cup A_3)|$$

$$= |A_1| + |A_2| + |A_3| - |A_2 \cap A_3|$$

$$- |(A_1 \cap A_2) \cup (A_1 \cap A_3)|$$

$$= |A_1| + |A_2| + |A_3| - |A_2 \cap A_3| - \big(|A_1 \cap A_2| + |A_1 \cap A_3| - |(A_1 \cap A_2) \cap (A_1 \cap A_3)|\big)$$

$$= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$