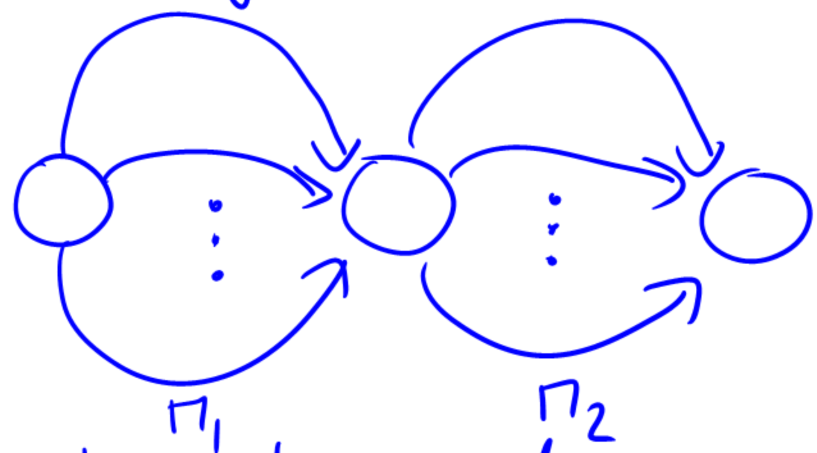
Chapter 5 Counting



Basic Counting Principles

Product rule and sum rule

The Product Rule

Procedure can be broken down into a sequence of two tasks.

Assume there are $\left\{ \begin{matrix} n_1 \\ n_2 \end{matrix} \right\}$ ways to do the $\left\{ \begin{matrix} \text{first} \\ \text{second} \end{matrix} \right\}$ task

Then there are $n_1 \cdot n_2$ ways to do the procedure.

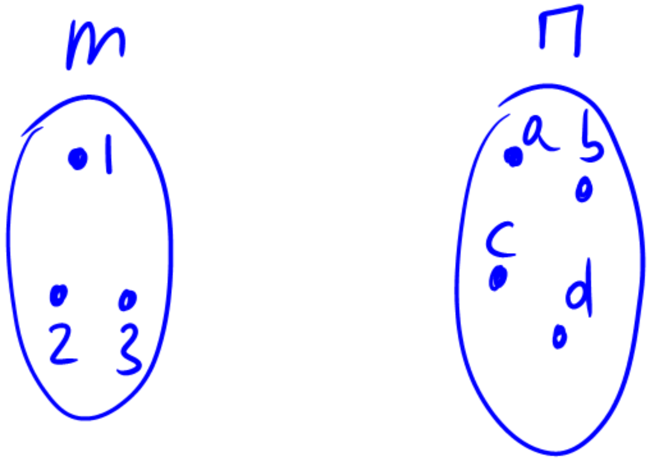
Counting Functions

$|A| = m \quad |B| = n$

How many function $f: A \to B$ are there?

$- m \cdot n$

$- 1$



n^m

$f(1) \quad f(2) \quad f(3)$

a b c d

Counting One-to-One Functions

injective

$f : A \to B$

$|A| = m \qquad |B| = n$

- $n < m \qquad 0$
- $n \geq m$

$$n(n-1)\cdots(n-m+1)$$

$$= \frac{n!}{(n-m)!} = \frac{n(n-1)\cdots(n-m+1)\cancel{(n-m)\cdots 1}}{\cancel{(n-m)\cdots 1}}$$

Counting Subsets of a Finite Set

$|A| = n$

$|P(A)| = 2^n$

$A = \{1, 2, 3, \ldots, n\}$

$B \subseteq A$

$|A| = m \quad |B| = n$

$|A \times B| = m \cdot n$

$|P(A \times B)| = 2^{m \cdot n} \neq 2^{n+m}$

	1	2	3	4
B	0	0	1	0
B	1	0	1	1

$n = 4$

$B = \{3\}$

$B = \{1, 3, 4\}$

The Sum Rule

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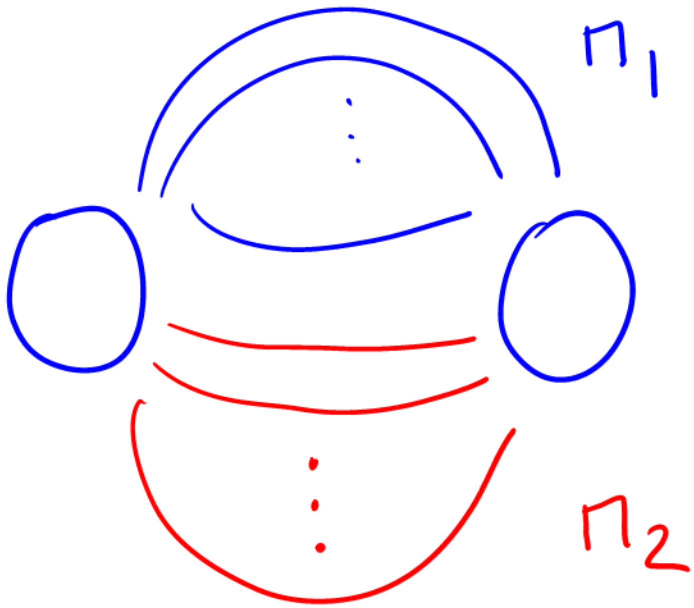


$n_1 + n_2$

A_1 and A_2 are disjoint

$A_1 \cap A_2 = \emptyset$

$|A_1 \cup A_2| = n_1 + n_2$

$|A_1| = n_1$

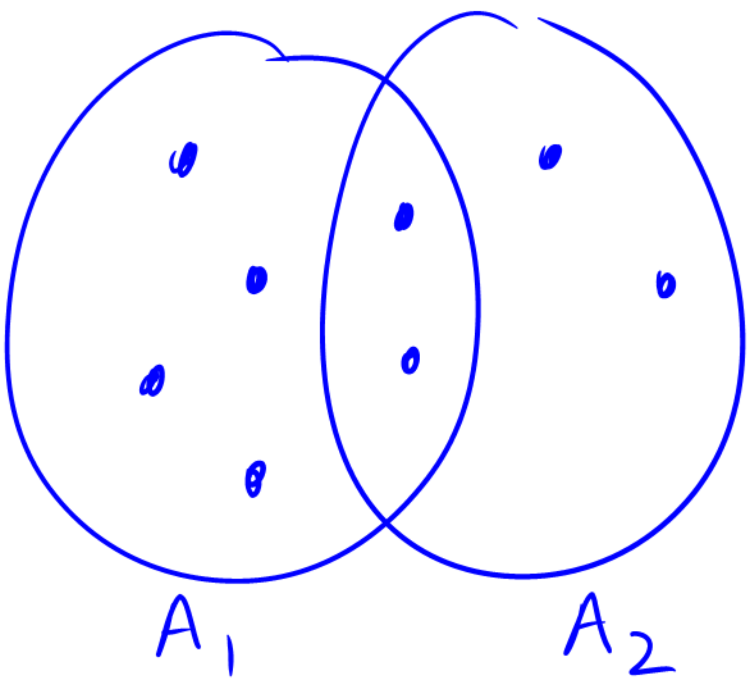
$|A_2| = n_2$

The Inclusion-Exclusion Principle

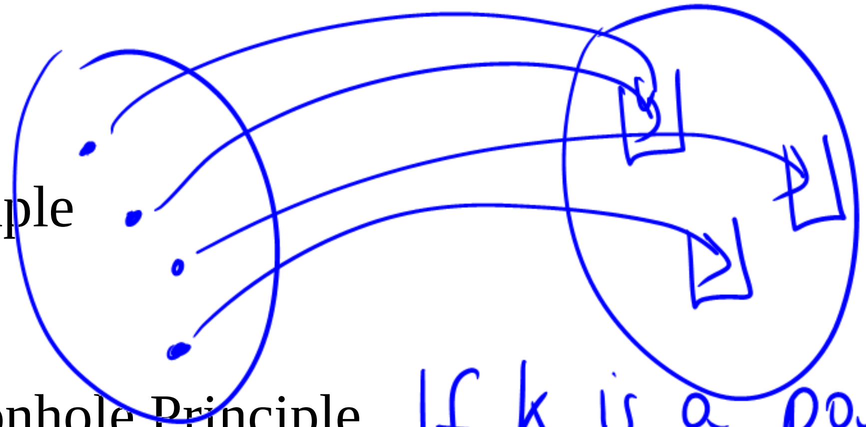
$A_1 \qquad A_2$

$$|A_1 \cup A_2| \leq |A_1| + |A_2|$$

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$



The Pigeonhole Principle



Theorem 1: The Pigeonhole Principle If k is a positive integer and $(k+1)$ or more objects are placed into k boxes, then there is at least one box containing two or more elements.

$|A| \geq k+1$ $B = |k|$

There cannot exist an injective function $f: A \to B$.

$N \geq k+1$ objects

k boxes

$\left\lceil \frac{N}{k} \right\rceil \geq \left\lceil \frac{k+1}{k} \right\rceil = 2$

The Generalized Pigeonhole Principle

Theorem 2: The Generalized Pigeonhole Principle

If N objects are placed into k boxes, then there is at least one box containing at least $\left\lceil \frac{N}{k} \right\rceil$ objects.

Proof by contradiction:

Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects. Then, the total number of objects

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = k\frac{N}{k} = N$$

Permutations and Combinations

Definition (Permutation):

Theorem 1:

Definition (Combination)

Theorem 2:

Corollary 2:

Definition 1 (Combinatorial Proof):

Binomial Coefficients

Theorem 1 (The Binomial Theorem):

Corollary 1:

Corollary 2:

Corollary 3:

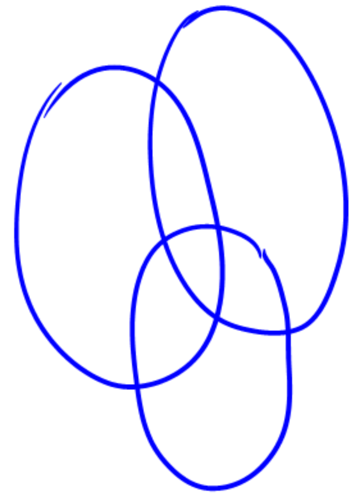
Pascal’s Identity and Triangle

Vandermonde's Identity

Generalized Permutations and Combinations

Permutations with Repetion

~~Combinations with Repetition~~



$$|A_1 \cup \underbrace{(A_2 \cup A_3)}_{B}| =$$

$$= |A_1| + |B| - |A_1 \cap B|$$

$$= |A_1| + |A_2 \cup A_3| - |A_1 \cap (A_2 \cup A_3)|$$

$$= |A_1| + |A_2| + |A_3| - |A_2 \cap A_3|$$

$$- |(A_1 \cap A_2) \cup (A_1 \cap A_3)|$$

$$= |A_1| + |A_2| + |A_3| - |A_2 \cap A_3| - \big(|A_1 \cap A_2| + |A_1 \cap A_3| - |(A_1 \cap A_2) \cap (A_1 \cap A_3)|\big)$$

$$= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$