### Intro to Discrete Structures Lecture 15

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### **DeMorgan for Intersection**

Example 10: Prove

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

whenever  $A_1, A_2, \ldots, A_n$  are subset of a universal U and  $n \ge 2$ .

### **DeMorgan for Intersection**

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### **Creative Uses of Mathematical Induction**

Example: Show that every  $2^n \times 2^n$  checkerboard with one square removed can be tiled using L-shaped triominoes.

## **Creative Uses of Mathematical Induction**

# **Strong Induction**

Strong induction is based on the rule of inference

1. 
$$P(1)$$
  
2.  $\forall k ( \wedge_{j=1}^{k} P(j) \rightarrow P(k+1))$   
3.  $\therefore \forall n P(n)$ 

which is true for the domain of natural numbers  $\mathbb{N}$ .

# **Strong Induction**

To prove that P(n) is true for all natural numbers n, where P(n) is a propositional function, we complete two steps:

- **•** Basis step: We verify that P(1) is true.
- We show that the conditional statement

$$\bigwedge_{j=1}^{k} P(j) \to P(k+1)$$

is true for all natural numbers k.

### **Existence of Prime Factorization**

Example: Show that if n is an integer greater than 1, then n can be written as the product of primes.

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Example 2: Show that if n is an integer greater than 1, then n can be written as the product of primes.

# Winning Strategy

Example 3:

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# **Well-Ordering Property**

The validity of both the principle of mathematical induction and strong induction follows from a fundamental axiom of set of integers, the **well-ordering property**.

THE WELL-ORDERING PROPERTY: Every nonempty set of nonnegative integers has a least element.

# **Division Algorithm**

Example 5: Use the well-ordering property to prove the division algorithm.

# **Division Algorithm**

# **Cycles in Round-Robin Tournament**

Example 6:

## **Cycles in Round-Robin Tournament**

### **Recursive Definitions & Structural Induction**

Sometimes it is difficult to define an object explicitly. However, it may be easy to define this object in terms of itself. This process is called recursion.

## **Recursively Defined Functions**

- We use two steps to define a function with the set of nonnegative integers as its domain:
  - Basis Step: Specify the value of the function at zero.
  - Recursive Step: Give a rule for finding its values from its values at smaller integers.
- Example 1: Suppose that is recursively defined by

$$f(0) = 3,$$
  
 $f(n+1) = 2f(n) + 3.$ 

Find f(1), f(2), f(3), and f(4).

#### **Fibonacci numbers**

Definition 1: The Fibonacci numbers  $f_0$ ,  $f_1$ ,  $f_2$ , ... are defined by the equation  $f_0$ ,  $f_1 = 1$ , and

$$f_n = f_{n+1} + f_{n+2}$$

for n = 2, 3, 4, ...

### **Growth of the Fibonacci Numbers**

Example 6: Show that whenever  $n \ge 3$ ,  $f_n > \alpha^{n-2}$ , where  $\alpha = (1 + \sqrt{5})/2$ .

### **Growth of the Fibonacci Numbers**

### **Recursively Defined Sets and Structures**

Example 7: Consider the subset *S* of integers defined by

**•** Basis step:  $3 \in S$ .

• Recursive step: If  $x \in S$  and  $y \in S$ , then  $x + y \in S$ .

### **Well-Formed Formulae**

Example 10: We can define the set of well-formed formulae for compound statement forms involving T, F, propositional variables, and operators from the set  $\{\neg, \land, \lor, \rightarrow, \leftrightarrow\}$ .

- Basis step: T, F, and, s, where s is a propositional variable, are well-formed formulae.
- Recursive step: If E and F are well-formed formulae, then (¬E), (E ∧ F), (E ∨ F), (E → F), and (E ↔ F) are well-formed.

## **Structural Induction**

A proof by structural induction consists of two parts:

- Show that the result holds for all elements in the basis step of the recursive definition.
- Show that if the statement is true for each of the elements used to construct new elements in the recursive step of the definition, the result holds for these new elements.
- The validity of structural induction follows from the principle of mathematical induction of nonnegative integers. To see this, let P(n) state that this claim is true for all elements that are generated by n or few applications of these rules.

### **Structural Induction**

Example 13: Show that every well-formed formulae for compound propositions contains an equal number of left and right parentheses.