Intro to Discrete Structures Lecture 15

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DeMorgan for Intersection

Example 10: Prove

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

whenever $A_1, A_2, \ldots, A_n$ are subset of a universal U and $n \geq 2$.

Basis step $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$

DeMorgan for Intersection

Inductive step

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

$$\overline{\bigcap_{j=1}^{n+1} A_j} = \overline{\bigcap_{j=1}^{n} A_j \cap A_{n+1}}$$

$$= \overline{\bigcap_{j=1}^{n} A_j} \cup \overline{A_{n+1}}$$

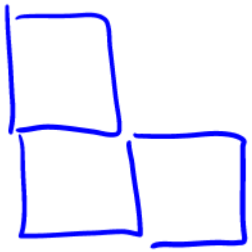
$$= \bigcup_{j=1}^{n} \overline{A_j} \cup \overline{A_{n+1}} = \bigcup_{j=1}^{n+1} \overline{A_j}$$

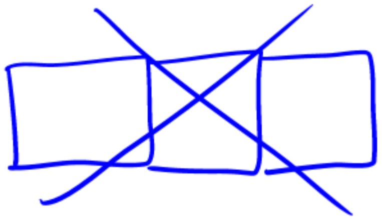
DeMorgan for Intersection

Creative Uses of Mathematical Induction

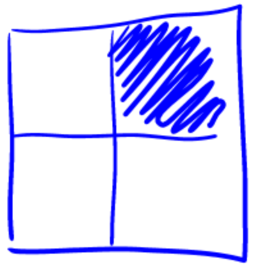
$n \geq 1$

Example: Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using L-shaped triominoes.

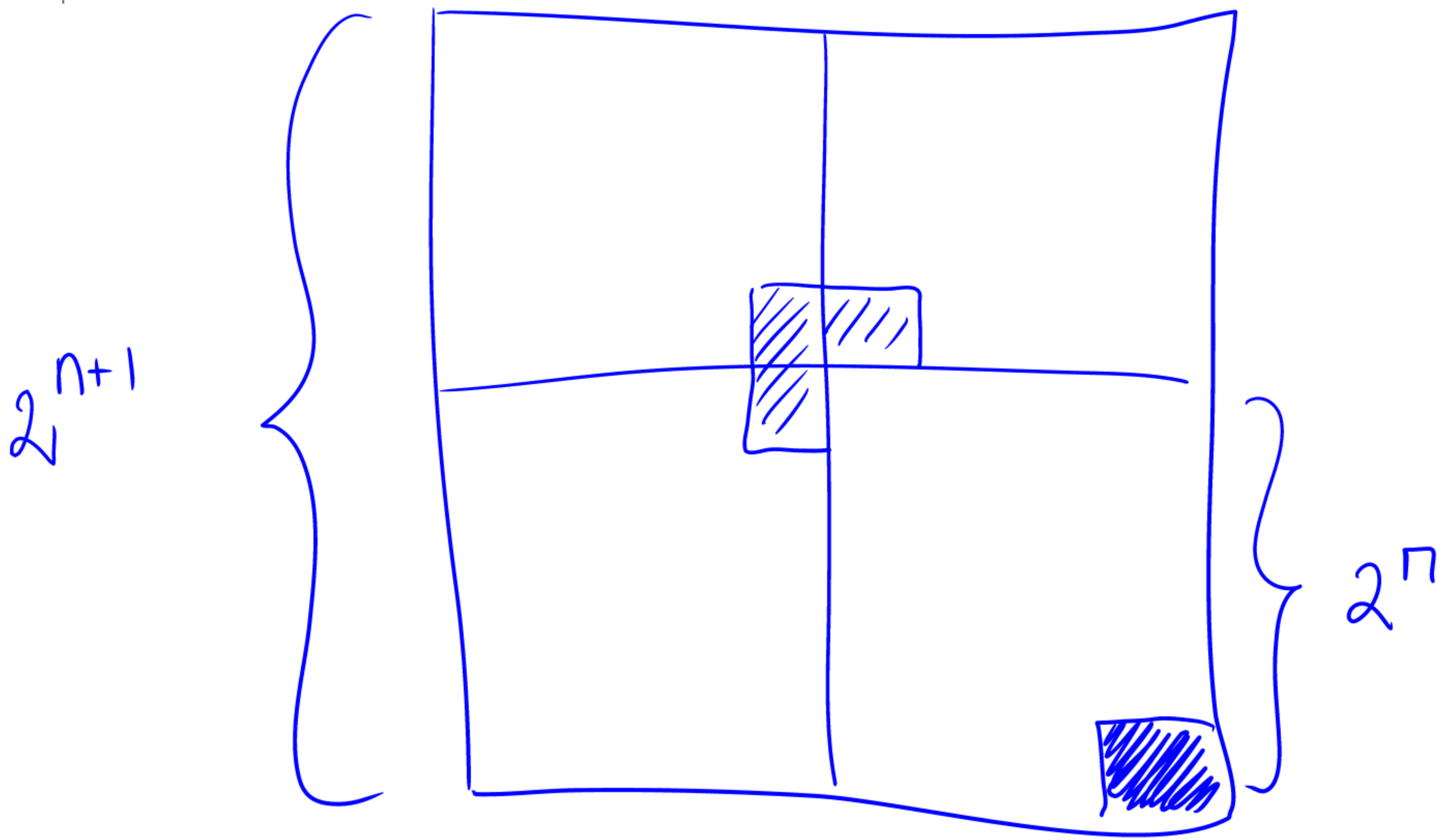




$n = 1$



Creative Uses of Mathematical Induction



Strong Induction

Strong induction is based on the rule of inference

$$
\begin{array}{lll}
1. & & P(1) \\
2. & & \forall k \, (\wedge_{j=1}^{k} P(j) \rightarrow P(k+1)) \\
\hline
3. & \therefore & \forall n P(n)
\end{array}
$$

which is true for the domain of natural numbers $\mathbb{N}$.

Strong Induction

To prove that $P(n)$ is true for all natural numbers n, where $P(n)$ is a propositional function, we complete two steps:

- Basis step: We verify that $P(1)$ is true.
- We show that the conditional statement

$$\bigwedge_{j=1}^{k} P(j) \rightarrow P(k+1)$$

is true for all natural numbers k.

Existence of Prime Factorization

Example: Show that if n is an integer greater than 1, then n can be written as the product of primes.

basis step

2 prime $\qquad 2 = 2$

we assume that any number less or equal to n can be written as a product of primes.

we assume that

$$\bigwedge_{j=2}^{n} P(j) \text{ is true}$$

Existence of Prime Factorization

$n+1$

$n+1$ prime $\Longrightarrow P(n+1) = T$

$n+1$ composite $\Longrightarrow$ there two integer a and b with $1<a,b<n$

s.t. $n+1 = a \cdot b$

Existence of Prime Factorization

Example 2: Show that if n is an integer greater than 1, then n can be written as the product of primes.

Winning Strategy

Example 3:

	left pile	right pile
$n =$	1	1
Player A	$\emptyset$	1
Player B	$\emptyset$	$\emptyset$

Winning Strategy

We assume that in the situation

left pile right pile

n n

Player B wins.

Winning Strategy

	left pile	right pile	
	$n+1$	$n+1$	
Player A	$\emptyset$	$n+1$	$\Rightarrow$ B wins
	$n+1$	$\emptyset$	$\Rightarrow$ B wins
Player A	$m \leq n$	$n+1$	
Player B	m	m	

Well-Ordering Property

The validity of both the principle of mathematical induction and strong induction follows from a fundamental axiom of set of integers, the **well-ordering property**.

THE WELL-ORDERING PROPERTY: Every nonempty set of nonnegative integers has a least element.

Division Algorithm

Example 5: Use the well-ordering property to prove the division algorithm.

Division Algorithm

Cycles in Round-Robin Tournament

Example 6:

Cycles in Round-Robin Tournament

Recursive Definitions & Structural Induction

Sometimes it is difficult to define an object explicitly. However, it may be easy to define this object in terms of itself. This process is called recursion.

Recursively Defined Functions

- We use two steps to define a function with the set of nonnegative integers as its domain:
  - Basis Step: Specify the value of the function at zero.
  - Recursive Step: Give a rule for finding its values from its values at smaller integers.

- Example 1: Suppose that is recursively defined by

$$\begin{aligned} f(0) &= 3\,, \\ f(n+1) &= 2f(n)+3\,. \end{aligned}$$

Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

Fibonacci numbers

Definition 1: The Fibonacci numbers f_0, f_1, f_2, . . . are defined by the equation f_0, $f_1 = 1$, and

$$f_n = f_{\underset{n-1}{\cancel{n+1}}} + f_{\underset{n-2}{\cancel{n+2}}}$$

for $n = 2, 3, 4, \ldots$.

1 1 2 3 5 8 13 . . .

Growth of the Fibonacci Numbers

Example 6: Show that whenever $n \geq 3$, $f_n > \alpha^{n-2}$, where $\alpha = (1 + \sqrt{5})/2$.

Growth of the Fibonacci Numbers

Recursively Defined Sets and Structures

Example 7: Consider the subset S of integers defined by

- Basis step: $3 \in S$.
- Recursive step: If $x \in S$ and $y \in S$, then $x + y \in S$.

Well-Formed Formulae

Example 10: We can define the set of well-formed formulae for compound statement forms involving T, F, propositional variables, and operators from the set $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$.

- Basis step: T, F, and, s, where s is a propositional variable, are well-formed formulae.
- Recursive step: If E and F are well-formed formulae, then $(\neg E)$, $(E \wedge F)$, $(E \vee F)$, $(E \rightarrow F)$, and $(E \leftrightarrow F)$ are well-formed.

$(\neg T)$

$\bigwedge_{j=1}^{n} A_j = A_1 \wedge A_2 \wedge \ldots \wedge A_n$

Structural Induction

- A proof by structural induction consists of two parts:
  - Show that the result holds for all elements in the basis step of the recursive definition.
  - Show that if the statement is true for each of the elements used to construct new elements in the recursive step of the definition, the result holds for these new elements.

- The validity of structural induction follows from the principle of mathematical induction of nonnegative integers. To see this, let $P(n)$ state that this claim is true for all elements that are generated by n or few applications of these rules.

Structural Induction

Example 13: Show that every well-formed formulae for compound propositions contains an equal number of left and right parentheses.

$$f(0) = 3$$
$$f(n+1) = 2f(n) + 3$$

$$f(1) = 2f(0) + 3$$
$$= 2 \cdot 3 + 3$$
$$= 9$$

$$f(2) = 2 \cdot 9 + 3$$
$$= 21$$