$$\sum_{i=1}^{50} \sum_{j=i}^{50} ij \qquad |c|$$

Intro to Discrete Structures
Lecture 14

Pawel M. Wocjan

School of Electrical Engineering and Computer Science

University of Central Florida

wocjan@eecs.ucf.edu

Mathematical Induction

Mathematical induction is based on the rule of inference

$$\begin{array}{lll} 1. & & P(1) \\ 2. & & \forall k\,(P(k) \rightarrow P(k+1)) \\ \hline 3. & \therefore & \forall n P(n) \end{array}$$

which is true for the domain of natural numbers $\mathbb{N}$.

Climbing an Infinite Ladder

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung of the ladder.
3. Therefore, we can reach any rung of the ladder.

Principle Mathematical Induction

To prove that $P(n)$ is true for all natural numbers n, where $P(n)$ is a propositional function, we complete two steps:

- Basis step: We verify that $P(1)$ is true.
- We show that the conditional statement

$$P(k) \to P(k+1)$$

is true for all natural numbers k.

Warning

- In a proof of mathematical induction is is **not** assumed that $P(k)$ is true for all k.
- It is only shown that **if it is assumed** that $P(k)$ is true, then $P(k+1)$ is also true.
- Thus, a proof of mathematical induction is not a case of begging the question, or circular reasoning.

Example

Example: Show that

$$P(n): \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Basis step

$$\sum_{i=1}^{1} i = 1 \quad = \quad \frac{1(1+1)}{2} = 1$$

Example

Let k be arbitrary. We have to show that $P(k) \to P(k+1)$ is true.

We assume that $P(k)$ is true.

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

We show to show that $P(k+1)$ is true.

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

$$\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

Example

Example 2: Conjecture a formula for the sum of the first n odd positive integers. Then prove your conjecture using mathematical induction.

$$\sum_{i=0}^{n-1}(2i+1)$$

$n=1$	1	1
$n=2$	$1+3$	$1+3$

$$= \left(\sum_{i=0}^{n-1} 2i\right) + \underbrace{\left(\sum_{i=0}^{n-1} 1\right)}_{n}$$

$$= 2\left(\sum_{i=0}^{n-1} i\right) + n$$

$$= 2\left(\sum_{i=1}^{n-1} i\right) + n$$

$$= 2\,\frac{(n-1)\overbrace{(n-1+1)}^{n}}{2} + n$$

$$= n^2 - n + n = n^2$$

Example

Example

Find a formula for

$$\sum_{j=1}^{n} (-1)^j j$$

assume that
n is even
$n = 2m$

||

$n=1$ -1

$n=2$ $-1+2$

$n=3$ $-1+2-3$

$n=4$ $-1+2-3+4$

$n=5$ $-1+2-3+4-5$

$n=6$ $-1+2-3+4-5+6$

$+1+2+3+4+5+6$

Example

$$\sum_{j=1}^{2m} (-1)^j j = \underbrace{\left(\sum_{j=1}^{2m} j\right)} - 2\sum_{j=0}^{m-1}(2j+1)$$

$$\frac{\cancel{2}m(2m+1)}{\cancel{2}} - 2m^2$$

$$= 2m^2 + m - 2m^2 = m$$

The Number of Subsets of a Finite Set

$|A| = n$

$|P(A)| = 2^n$

$P(\emptyset) = 1$ ✓

$P(A) = 2^n$ for $|A| = n$ $A = \{1, 2, \ldots, n\}$

$B = A \cup \{n+1\}$ $|B| = |A| + 1 = n+1$

The Number of Subsets of a Finite Set

we have to look at the subsets of B

let C be a subset of B.

Either $n+1 \notin C$ or $n+1 \in C$

$\uparrow$ $\# = 2^n$

$\uparrow$ $\# 2^n$

DeMorgan for Intersection

Example 10: Prove

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

whenever $A_1, A_2, \ldots, A_n$ are subset of a universal U and $n \geq 2$.

DeMorgan for Intersection

DeMorgan for Intersection

Creative Uses of Mathematical Induction

Example: Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using L-shaped triominoes.

Creative Uses of Mathematical Induction

Strong Induction

Strong induction is based on the rule of inference

$$
\begin{array}{lll}
1. & & P(1) \\
2. & & \forall k\,(\wedge_{j=1}^{k} P(j) \rightarrow P(k+1)) \\
\hline
3. & \therefore & \forall n P(n)
\end{array}
$$

which is true for the domain of natural numbers $\mathbb{N}$.

Strong Induction

To prove that $P(n)$ is true for all natural numbers n, where $P(n)$ is a propositional function, we complete two steps:

- Basis step: We verify that $P(1)$ is true.
- We show that the conditional statement

$$\bigwedge_{j=1}^{k} P(j) \rightarrow P(k+1)$$

is true for all natural numbers k.

Existence of Prime Factorization

Example: Show that if n is an integer greater than 1, then n can be written as the product of primes.

Existence of Prime Factorization

Existence of Prime Factorization

Example 2: Show that if n is an integer greater than 1, then n can be written as the product of primes.

Winning Strategy

Example 3:

Winning Strategy

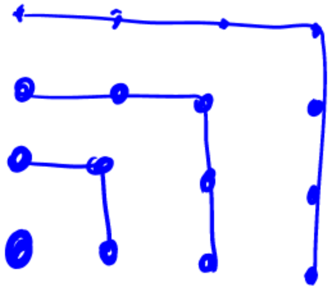
Winning Strategy

Universal generalization

- $Q(c)$ hold for an arbitrary c from the domain

$\therefore \forall c\, Q(c)$

$\forall k (P(k) \rightarrow P(k+1))$



1 3 5 7

$P(n): \sum_{i=0}^{n-1} (2i+1) = n^2$ is correct

Conjecture: $\sum_{i=0}^{n-1} (2i+1) = n^2$

Basis step ✓

We assume $\sum_{i=0}^{k-1} (2i+1) = k^2$ holds.

We have to show that $\sum_{i=0}^{k} (2i+1) = (k+1)^2$

$$\sum_{i=0}^{k} (2i+1) = (k+1)^2$$

$$\sum_{i=0}^{k} (2i+1) = \left[\sum_{i=0}^{k-1} (2i+1)\right] + 2k+1$$

$$= k^2 + 2k + 1$$

$$= (k+1)^2$$

$B = A \cup \{x\} \qquad x \notin A$

$$\mathcal{P}(B) = \mathcal{P}(A) \cup \qquad B \setminus \{x\}$$

$$\left\{ S \cup \{x\} \mid S \in \mathcal{P}(A) \right\}$$

$A = \{1, 2\}$ $B = \{1, 2, 3\}$

$\mathcal{P}(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

$\{S \cup \{3\} \mid S \in \mathcal{P}(A)\}$

$= \{\phi \cup \{3\}, \{1\} \cup \{3\}, \{2\} \cup \{3\}, \{1, 2\} \cup \{3\}\}$

$= \{\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$