Intro to Discrete Structures
Lecture 11

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Factorial Function

The **factorial function** $f : \mathbb{N} \rightarrow \mathbb{Z}^+$ is denoted by $f(n) = n!$.

The value of $f(n) = n!$ is the product of the first n positive integers, so

$$n! = 1 \cdot 2 \cdots (n-1) \cdot n$$

and $0! = 1$.

(Strictly) Increasing / Decreasing

Definition 6: Let $f : A \to B$ with $A, B \subseteq \mathbb{R}$.

The function f is called

- **increasing** if

$$f(x) \leq f(y)$$

- **strictly increasing** if

$$f(x) < f(y)$$

whenever $x < y$.

Addition / Multiplication of Functions

Definition 3: Let $f_1, f_2 : A \to \mathbb{R}$.

Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbb{R}$ defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)\,, \quad (1)$$

$$(f_1 f_2)(x) = f_1(x) f_2(x)\,. \quad (2)$$

Sequences and Summations

Sequences

Definition 1: A **sequence** is a function from a subset of the set of integers (usually the set $\{0, 1, 2, \ldots\}$ or the set $\{1, 2, 3, \ldots,\}$) to a set S.

We use the notation a_n to denote the image of the integer n.

We call a_n a **term** of the sequence.

Example 1: Consider the sequence $\{a_n\}$ with $a_n = 1/n$.

Geometric Progression

Definition 2: A **geometric progression** is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the **initial term** a and the **common ratio** r are real numbers.

Remark: A geometric progression is a discrete analogue of the exponential function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto ar^x$.

Arithmetic Progression

Definition 2: An **arithmetic progression** is a sequence of the form

$$a, a+d, a+2d, \ldots, a+nd, \ldots$$

where the **initial term** a and the **common difference** r are real numbers.

Remark: A geometric progression is a discrete analogue of the linear function $f : \mathbb{R} \to \mathbb{R},\ x \mapsto dx + a$.

The Tower of Hanoi

We are given a tower of n discs, initially stacked in decreasing size on one of three pegs:

The objective is to transfer the entire tower to one of the other pegs, moving only one disc at a time and never moving a larger one onto a smaller.

Find a simple expression for T_n, the number of minimal moves required to accomplish this.

Slicing Pizza

How many slices of pizza can a person obtain by making n straight cuts with a pizza knife?

Or, more academically: What is the maximum number of L_n of regions defined by n lines in the plane?

Find a simple expression for L_n.

The Josephus Problem

We start with n people numbered 1 to n around a circle, and we eliminate every **second** remaining person until only one survives. Denote this person by J_n. For example, here's the starting configuration for $n = 10$.

The elimination order is $2, 4, 6, 8, 10, 3, 7, 1, 9$, so $J_{10} = 5$ survives.

Solution to the Josephus Problem

Find a (closed form) formula for a_n that makes it possible to efficiently compute J_n for large n.

Solution to the Josephus Problem

Find a (closed form) formula for J_n that makes it possible to efficiently compute J_n for large n.

Every $n \in \mathbb{N}$ can be uniquely written as

$$2^m + \ell \text{ with } m \geq 0 \text{ and } 0 \leq \ell < 2^m.$$

Observe that $m = \lfloor \log_2 n \rfloor$ and $\ell = n - 2^m$.

The solution is

$$J_n = 2\ell + 1\,.$$

Summation

We now introduce **summation notation**. We begin by describing the notation used to express the sum of the terms

$$a_m, a_{m+1}, \ldots, a_n \ ,$$

from the sequence $\{a_n\}$.

We use the notation

$$\sum_{j=m}^{n} a_j \ , \quad \textstyle\sum_{j=m}^{n} a_j, \quad \text{or} \quad \textstyle\sum_{1 \le j \le n} a_j$$

to represent $a_m + a_{m+1} + \ldots + a_n$.

Summation

$$\sum_{j=m}^{n} a_j$$

Here the variable j is called the **index of summation**, and the choice of letter as the variable is arbitrary.

The index of summation runs through all integers starting with its **lower limit** m and ending with its **upper limit** n.

A large upper case Greek letter sigma, $\sum$, is used to denote the summation.

Geometric sums

Theorem 1: If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{a^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

Cardinality

Definition 4: The sets A and B have the same **cardinality** iff there is a bijection from A to B.

Countable vs. Uncountable

Definition 5: A set that either finite or has the same cardinality as the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ is called **countable**.

A set that is not countable is called **uncountable**.

The cardinality of a finite set S is denoted by $|S|$.

When an infinite set S is countable, we denote the cardinality of S by $\aleph_0$. We write $|S| = \aleph_0$ and say that S has cardinality "aleph null".

Odd Numbers

Example 18: Show that the set of odd natural numbers is a countable set.

Integers

Example 19: Show that the set $\mathbb{Z}$ of all integers is countable.

Rational Numbers

Example 19: Show that the set $\mathbb{Q}^+$ of positive rational numbers is countable.

Real Numbers

Example 21: Show that the set $\mathbb{R}$ of real numbers is uncountable.
Assume the contrary. Then, there is a bijection $\mathbb{N} \to [0, 1)$

$$
\begin{aligned}
r_1 &= d_{11}d_{12}d_{13}d_{14}\ldots \\
r_2 &= d_{21}d_{22}d_{23}d_{24}\ldots \\
r_3 &= d_{31}d_{32}d_{33}d_{34}\ldots \\
r_4 &= d_{41}d_{42}d_{43}d_{44}\ldots \\
&\vdots
\end{aligned}
$$

The decimal expansion $r_i = \sum_{j=1}^{\infty} d_{ij}/10^j$ is unique provided we exclude that we disallows infinite tails $999\ldots$ in the expansions.

Real Numbers

Cantor Diagonalization Argument

Form a new real number with decimal expansion $r = 0.d_1 d_2 d_3 d_4 \ldots$ where the decimal digits are determined by the following rule:

$$d_i = \begin{cases} 4 & \text{if } d_i i \neq 4 \\ 5 & \text{if } d_i i = 4 \end{cases}$$

Halting Problem

In computability theory, the halting problem is a decision problem which can be stated as follows: given a description of a program, decide whether the program finishes running or will run forever. This is equivalent to the problem of deciding, given a program and an input, whether the program will eventually halt when run with that input, or will run forever.

Alan Turing proved in 1936 that a general algorithm to solve the halting problem for all possible program-input pairs cannot exist.

Gödel's Incompleteness Theorem

Gödel's incompleteness theorems are two theorems of mathematical logic that establish inherent limitations of all but the most trivial axiomatic systems for mathematics. The theorems, proven by Kurt Gödel in 1931, are important both in mathematical logic and in the philosophy of mathematics.

The first incompleteness theorem states that no consistent system of axioms whose theorems can be listed by an "effective procedure" (essentially, a computer program) is capable of proving all facts about the natural numbers. For any such system, there will always be statements about the natural numbers that are true, but that are unprovable within the system.

Book

- *Gödel, Escher, Bach* by D. R. Hofstadter