Intro to Discrete Structures
Lecture 10

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Membership Table

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Generalized Union

Definition 6: The **union** of a collection of sets is the set that contains those element that are members of at least one set in the collection.

We use the notation

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$$

to denote the union of the sets $A_1, A_2, \ldots, A_n$.

$$x \in \bigcup_{i=1}^{n} A_i \Longleftrightarrow x \in A_1 \vee x \in A_2 \vee \ldots \vee x \in A_n$$

Generalized Intersection

Definition 6: The **intersection** of a collection of sets is the set that contains those element that are members of all sets in the collection.

We use the notation

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$$

to denote the intersection of the sets $A_1, A_2, \ldots, A_n$.

$$x \in \bigcap_{i=1}^{n} A_i \iff x \in A_1 \wedge x \in A_2 \wedge \ldots \wedge x \in A_n$$

Functions

Definition 1: Let A and B be nonempty sets.

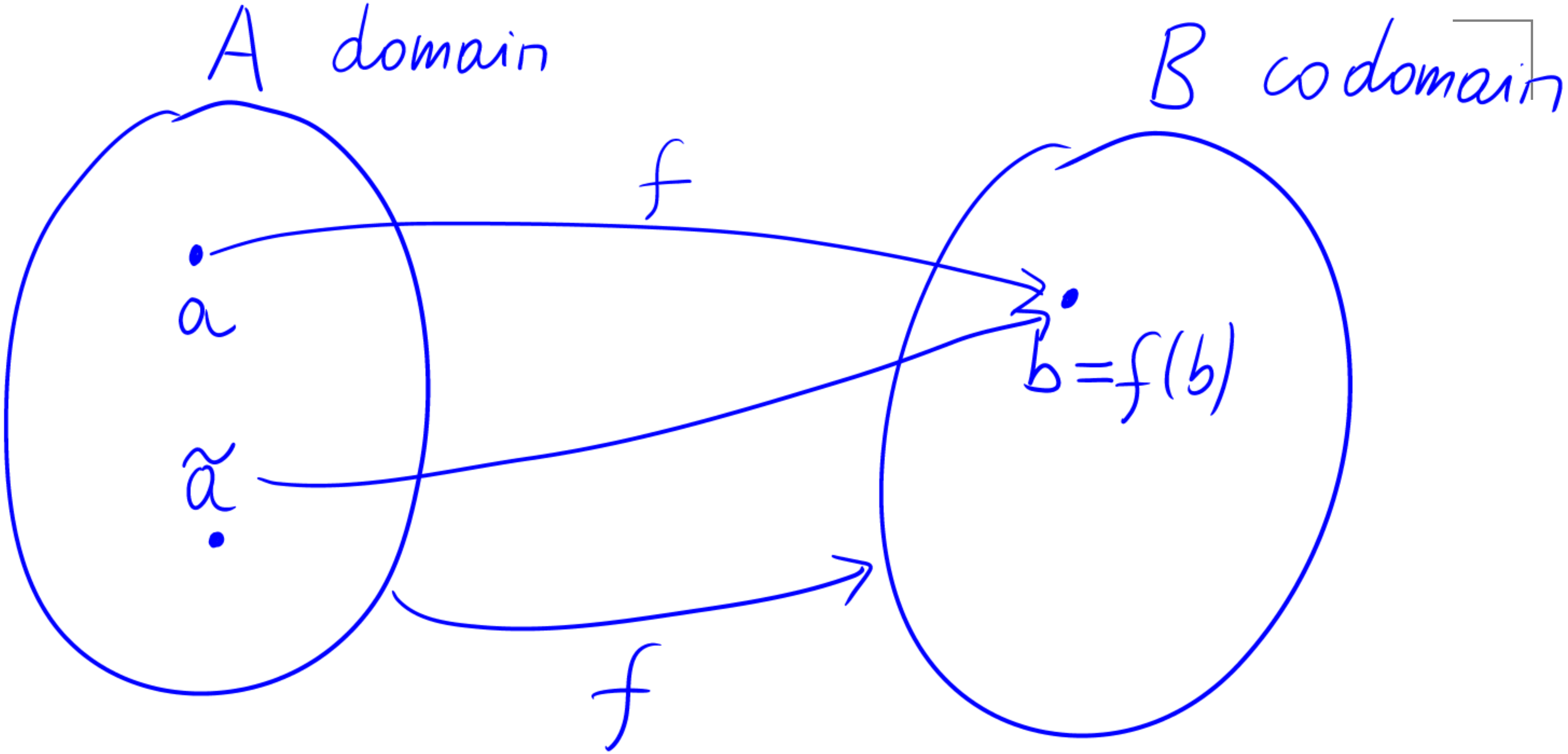
A **function** f from A to B is an assignment of exactly one element of B to each element A.

We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

We write $f : A \to B$ if f is a function from A to B.

sin(x)

Functions



b is the image of a
a is a preimage of b

Domain / Codomain

Definition 2: If $f:A \to B$, we say that A is the **domain** and B the **codomain** of F.

If $f(a) = b$, we say that b is the **image** of a and b is the preimage of $\cancel{b}$. a

The **range** of f is the set of all images of elements of A.

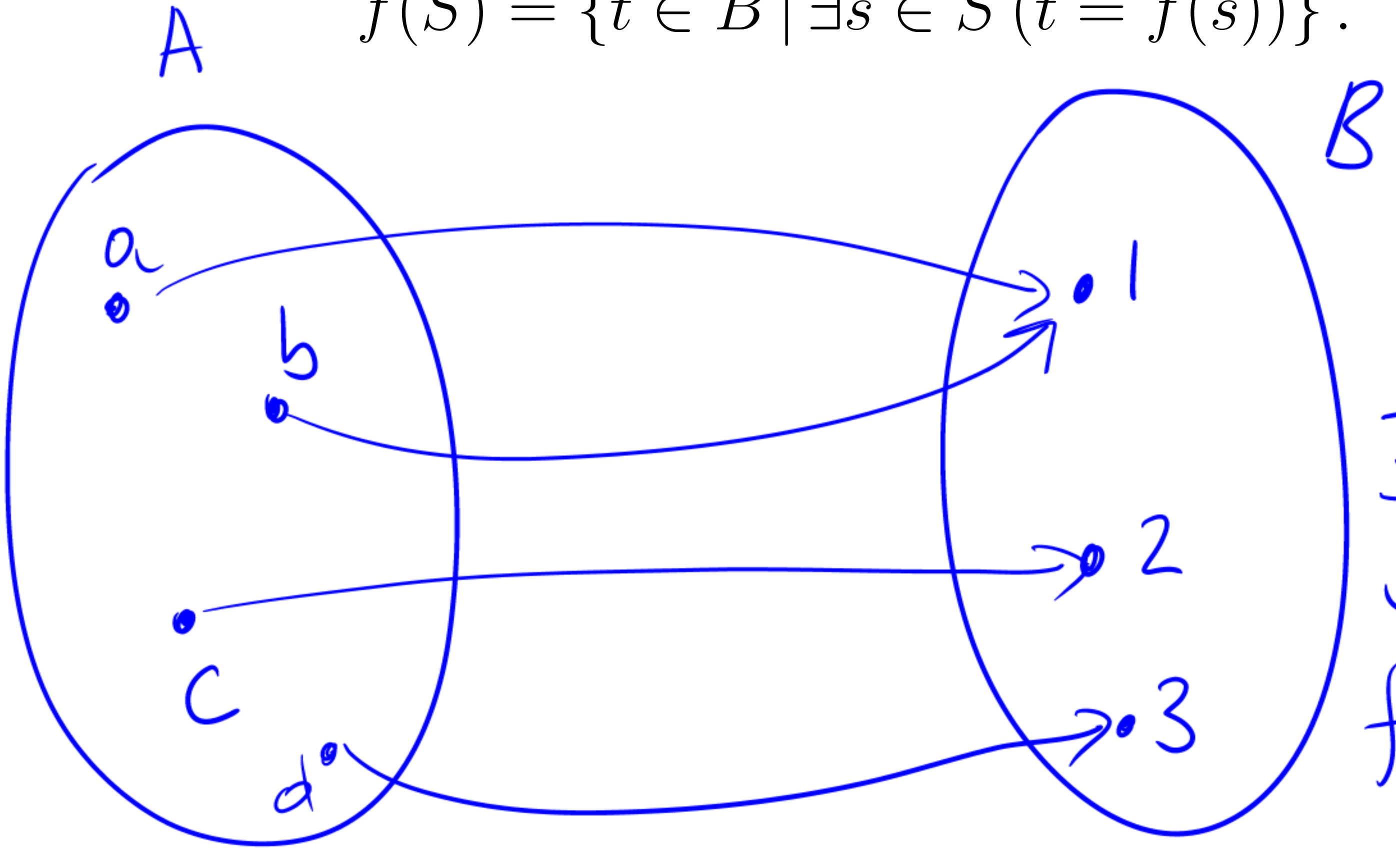
If $f:A \to B$, we also say that f **maps** from A to B.

Image

Definition: 4 Let $S \subseteq A$ and $f : A \to B$. The **image** of S under the function f is the set

1 a $1 = f(a)$

$$f(S) = \{t \in B \mid \exists s \in S\,(t = f(s))\}\,.$$



$S = \{a, b\}$

$f(S) = \{1\}$

$S = \{a, b, c\}$

$f(S) = \{1, 2\}$

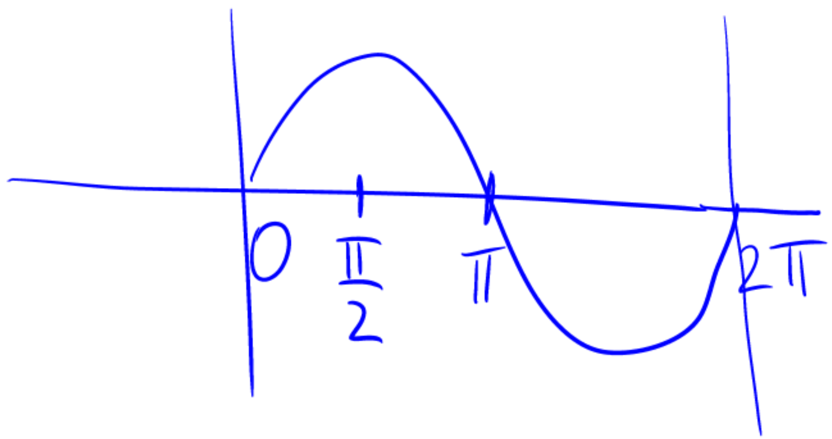
Injective Functions

Definition 5: The function $f : A \to B$ is called **injective** or **one-to-one** iff

$$\forall a \in A \, \forall b \in A \, (f(a) = f(b) \to a = b) \, .$$

$\sin : \mathbb{R} \to [-1, 1]$ not injective

$\sin : [0, \frac{\pi}{2}) \to [-1, 1]$ injective



Surjective Functions

Definition 7: The function $f : A \to B$ is called **surjective** or **onto** iff

$$\forall b \in B \, \exists a \in A \, (f(a) = b)) \, .$$

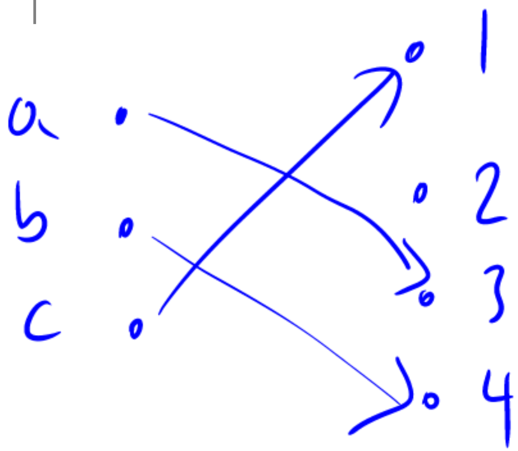
$\sin : \mathbb{R} \to \mathbb{R}$ not surjective

$\sin : \mathbb{R} \to [-1, 1]$ surjective

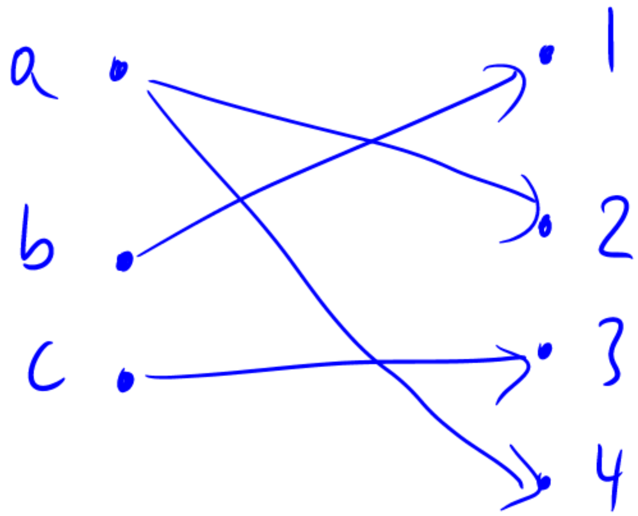
Bijective Function

Definition 8: The function $f : A \longrightarrow B$ is called **bijective** if it is both surjective and injective.

Examples



injective
not surjective

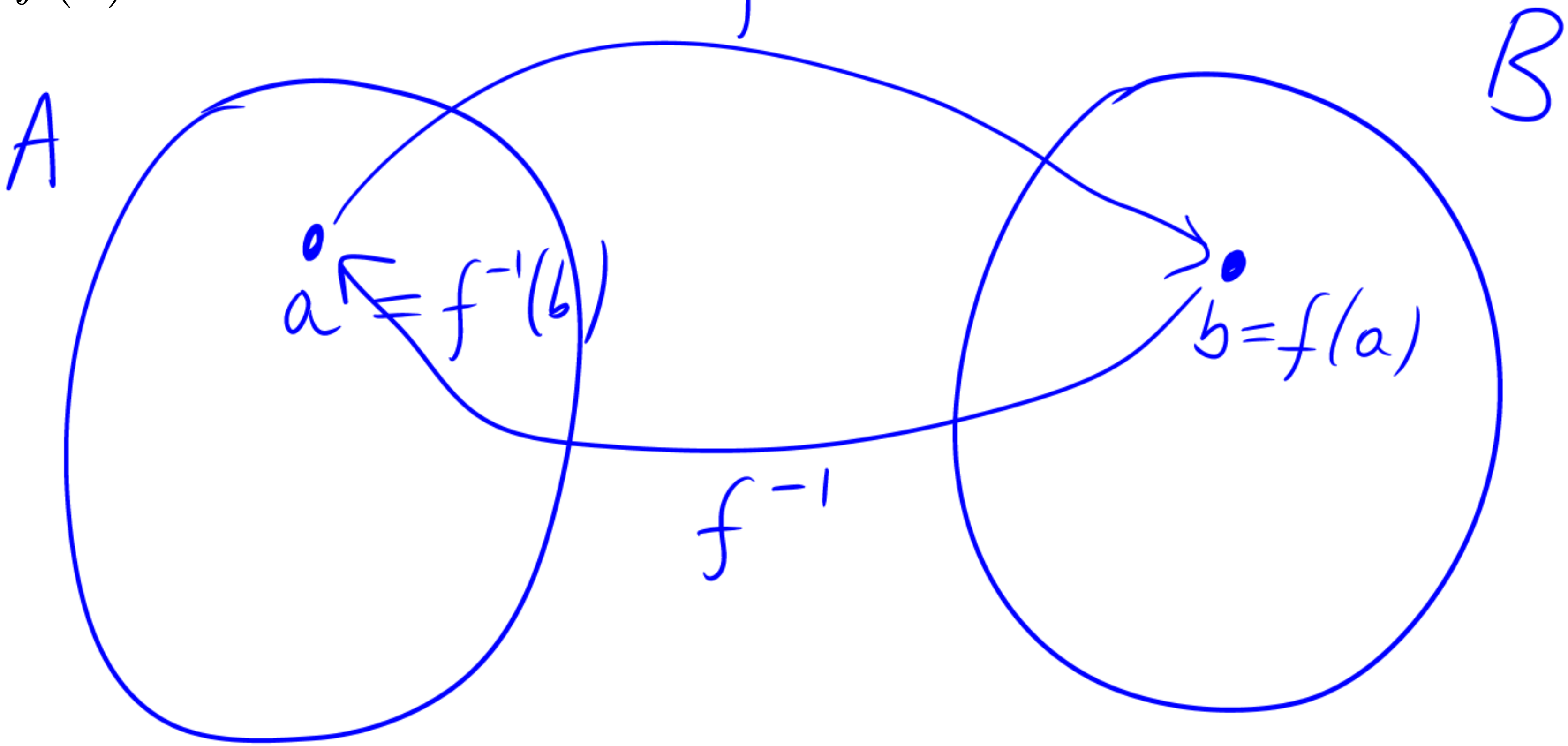


not a function

Inverse Function

Definition 9: Let $f : A \to B$ be a bijective function.

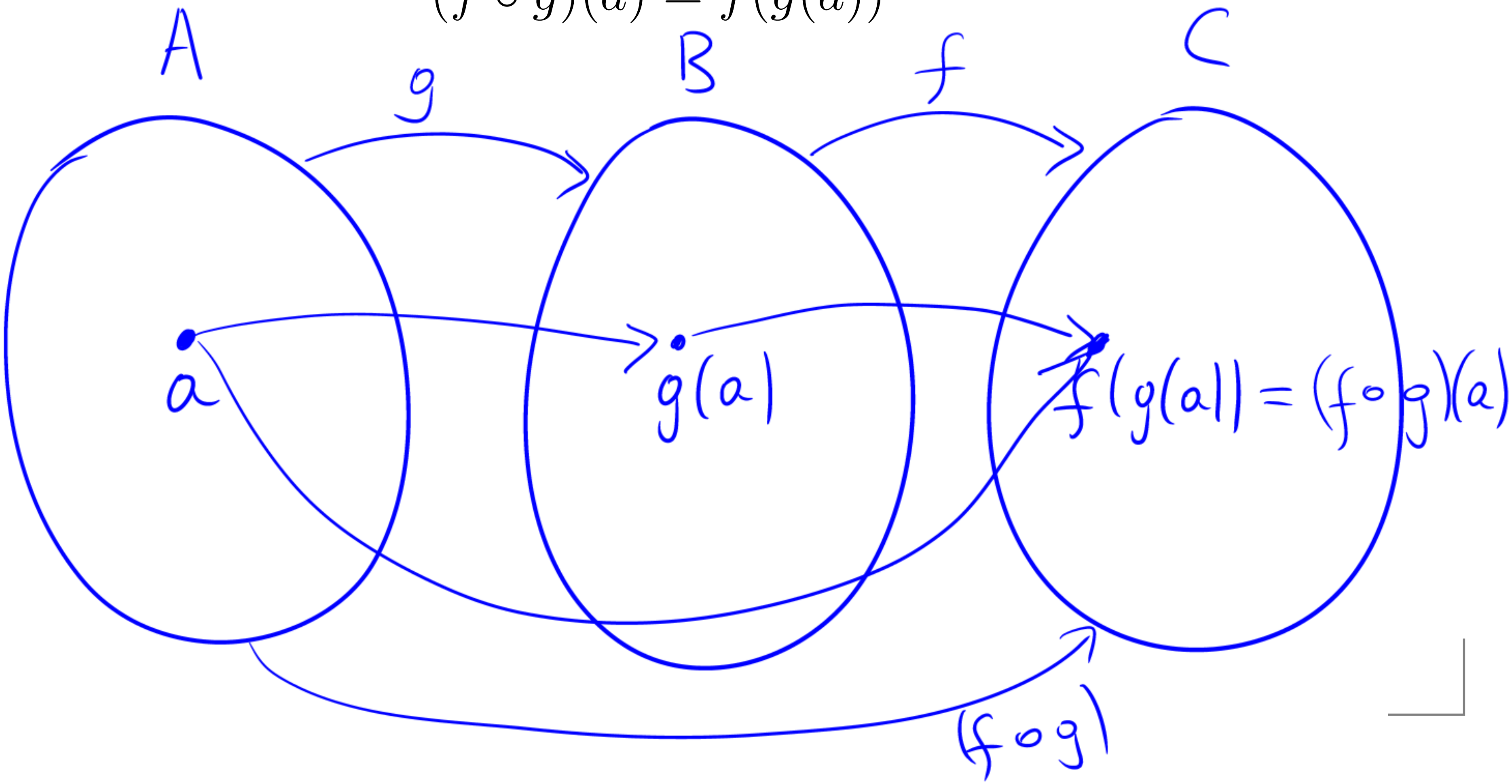
Then, the **inverse function** f^{-1} is the function from B to A that assigns to $b \in B$ the unique element $a \in A$ such that $f(a) = b$.



Composition of Functions

Let $g : A \rightarrow B$ and $f : B \rightarrow C$. The **composition** of the functions f and g, denoted by $(f \circ g)$, is defined by

$$(f \circ g)(a) = f(g(a))$$



Composition of f and f^{-1}

$f : A \to B$ bijective

$id_A : \begin{cases} A \to A \\ a \mapsto a \end{cases}$ $id_B : \begin{cases} B \to B \\ b \mapsto b \end{cases}$

$f \circ f^{-1} : \begin{cases} B \to B \\ b \mapsto b \end{cases}$ $f^{-1} \circ f : \begin{cases} A \to A \\ a \mapsto a \end{cases}$

$(f \circ f^{-1})(b) = b$ $(f^{-1} \circ f)(a) = a$

$f \circ f^{-1} = id_B$ $= f^{-1}(f(a))$

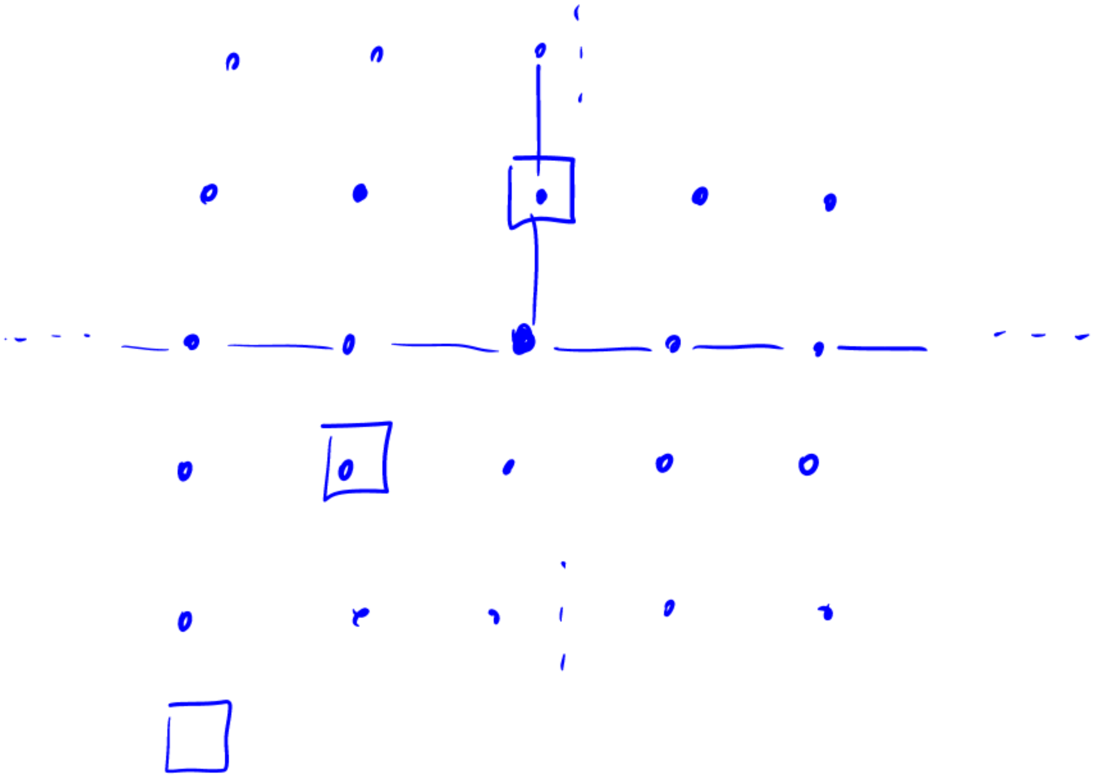
Graphs of Functions

Let $f : A \longrightarrow B$. The **graph of the function** f is the subset of $A \times B$ defined by

$$\{(a, b) \in A \times B \mid f(a) = b\}.$$

Examples

The graph of $f(n) = 2n + 1$ from $\mathbb{Z}$ to $\mathbb{Z}$.

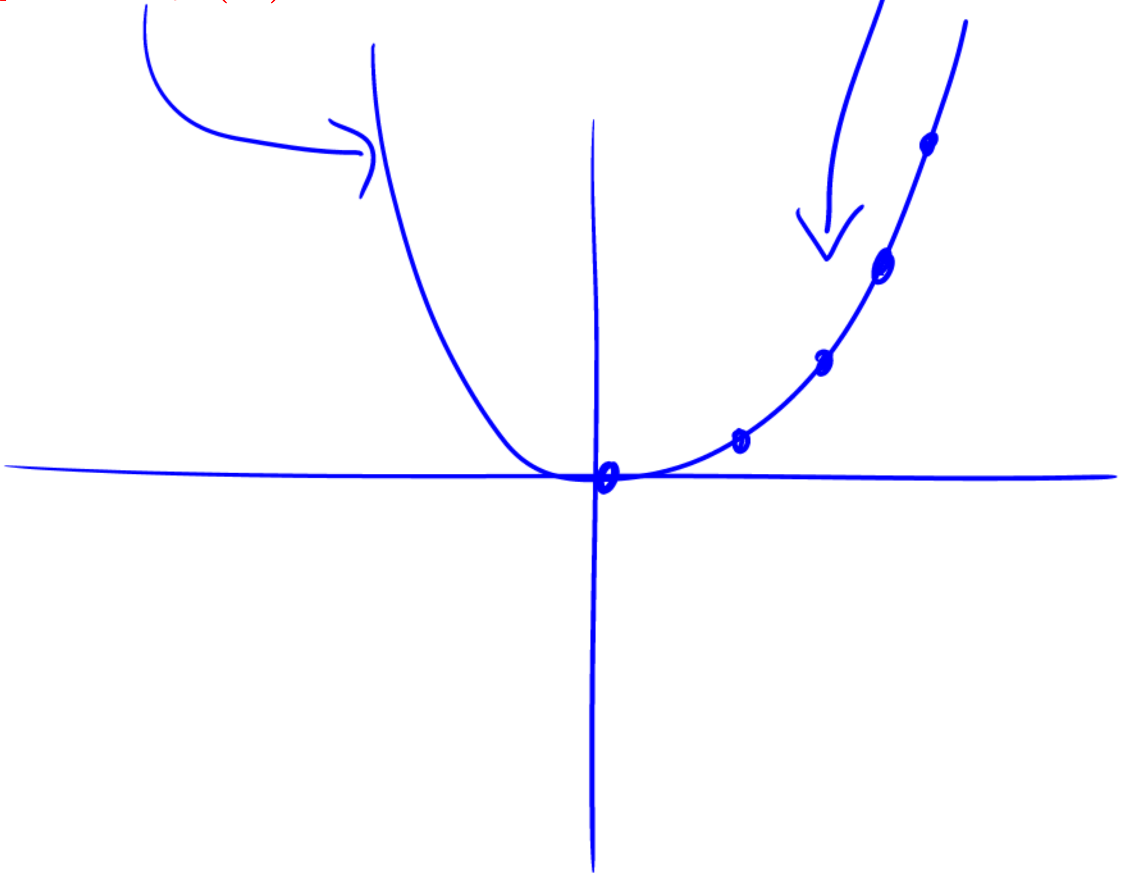


$\mathbb{Z} \times \mathbb{Z}$

Examples

The graph of $f(x) = x^2$ from $\mathbb{Z}$ to $\mathbb{Z}$. discrete points

The graph of $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}$.



Floor and Ceiling Functions

Definition 12: The **floor function** assigns to the real number x the largest integer that is less or equal to x.

The value of the floor function is denoted by $\lfloor x \rfloor$.

$\lfloor e^{\pi^{\sqrt{2}}} \rfloor =$

The **ceiling function** assigns to the real number x the smallest integer that is greater or equal to x.

The value of the ceiling function is denoted by $\lceil x \rceil$.

$\lfloor \pi \rfloor = 3$ $\lceil \pi \rceil = 4$

$\lfloor e \rfloor = 2$ $\lceil \sqrt{2} \rceil = 2$

Functions Graphs of Floor and Ceiling

Factorial Function

The **factorial function** $f : \mathbb{N} \rightarrow \mathbb{Z}^+$ is denoted by $f(n) = n!$.

The value of $f(n) = n!$ is the product of the first n positive integers, so

$$n! = 1 \cdot 2 \cdots (n-1) \cdot n$$

and $f(0) = 1$.

(Strictly) Increasing / Decreasing

Definition 6: Let $f : A \to B$ with $A, B \subseteq \mathbb{R}$.

The function f is called

- **increasing** if

$$f(x) \leq f(y)$$

- **strictly increasing** if

$$f(x) < f(y)$$

whenever $x < y$.

Addition / Multiplication of Functions

Definition 3: Let $f_1, f_2 : A \to \mathbb{R}$.

Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbb{R}$ defined by

$$\begin{aligned} (f_1 + f_2)(x) &= f_1(x) + f_2(x)\,, && (1) \\ (f_1 f_2)(x) &= f_1(x) f_2(x)\,. && (2) \end{aligned}$$

Sequences and Summations

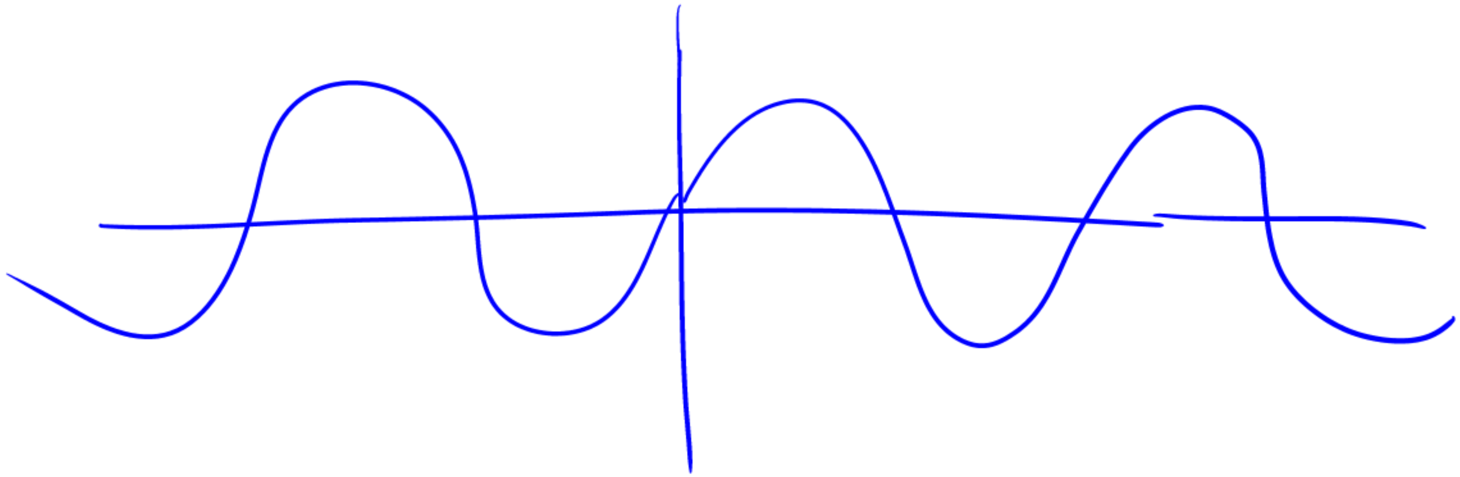
Sequences

Summation

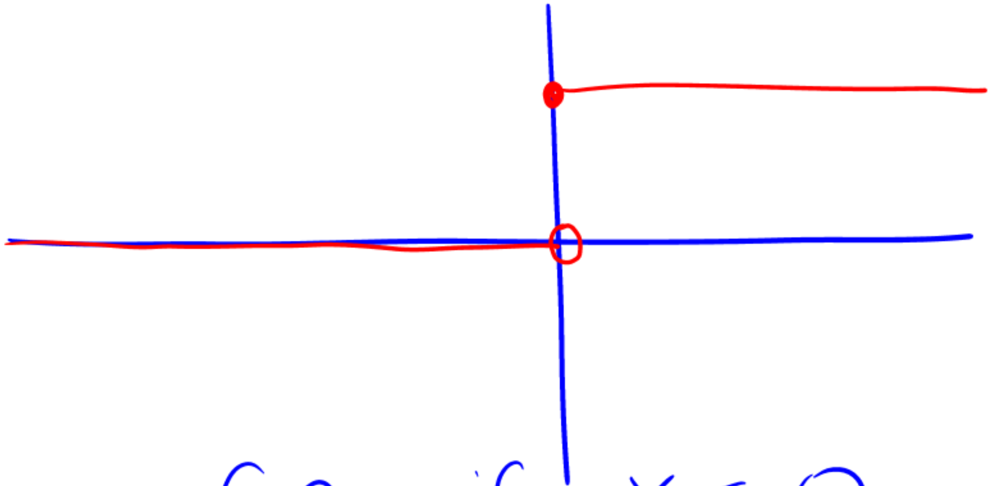
$\mathbb{R}$

$\sin : \mathbb{R} \to \mathbb{R}$

$\sin(\mathbb{R}) = [-1, 1]$



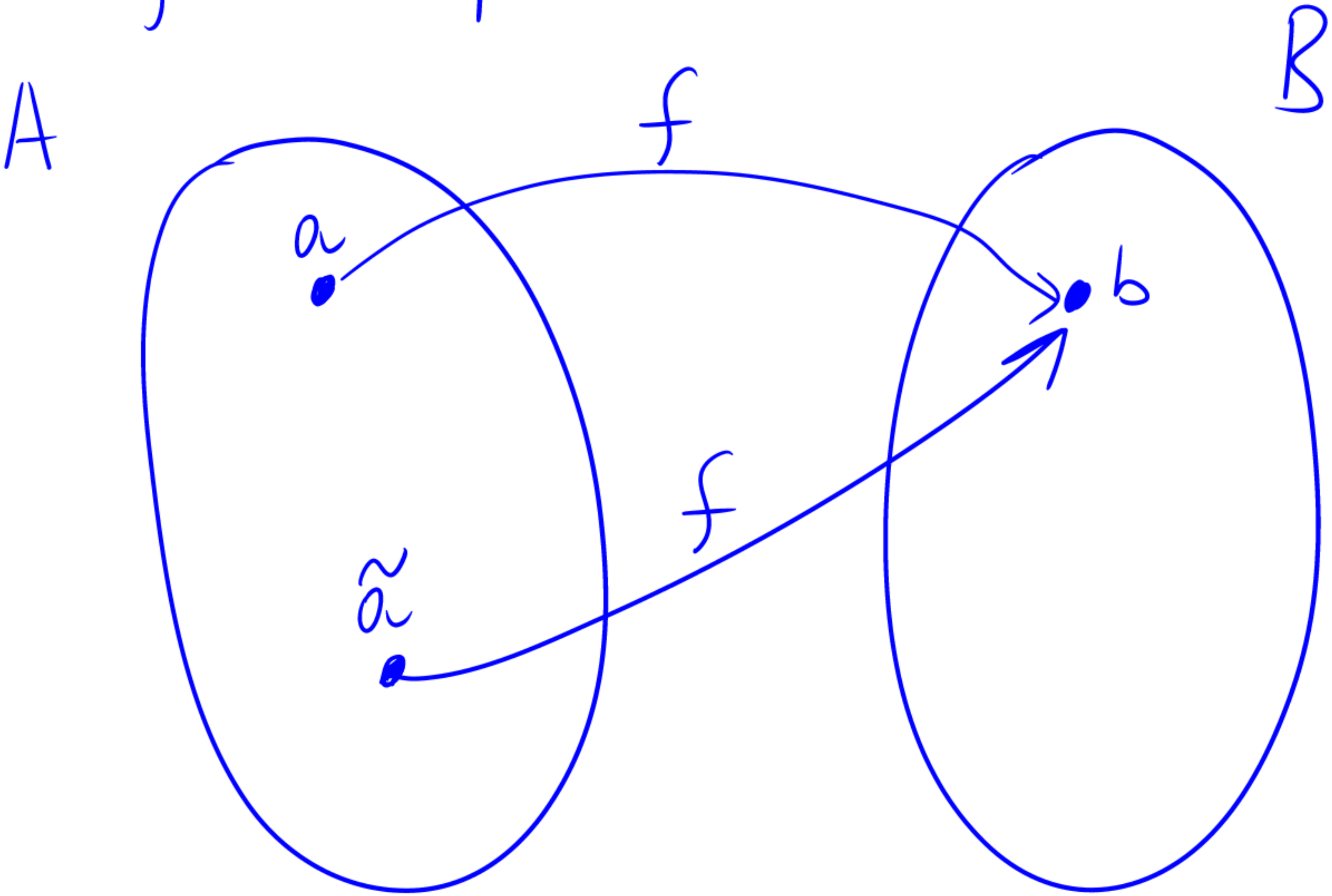
Step function



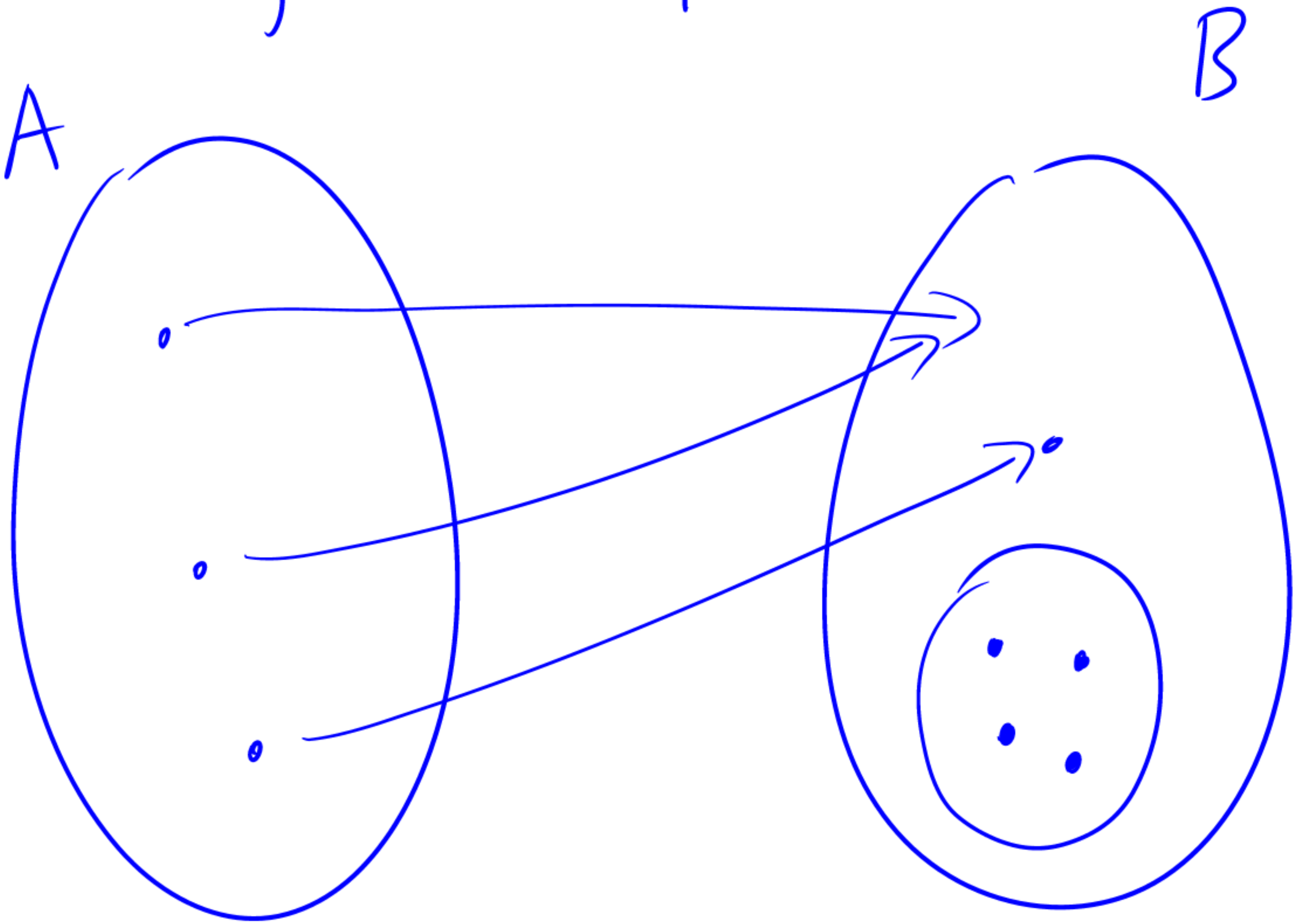
$$\text{step}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

$$\text{step}(\mathbb{R}) = \{0, 1\} \neq [0, 1]$$

f not injective



f not surjective



Square

$$\begin{cases} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto x^2 \end{cases}$$

Sin

$$\sin : \begin{cases} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto \sin(x) \end{cases}$$

$$f : \begin{cases} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto x+1 \end{cases}$$