

Intro to Discrete Structures

Lecture 8

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Proof Strategy

- We have seen two important methods for proving theorems of the form $\forall x(P(x) \rightarrow Q(x))$.

These two methods are

- the direct proof and
- the indirect proof

methods.

- It takes some practice (solving homework problems) to learn to recognize quickly the correct approach.
- Try first the direct approach. If it does not work then try the indirect approach.

Direct or Indirect Proof?

Definition: The real number r is **rational** if there exist integers p and q with $q \neq 0$ such that $r = p/q$.

Example 7: Prove that the sum of two rational numbers is rational.

Direct or Indirect Proof?

Example 8: Prove that if n is an integer and n^2 is odd, then n is odd.

Proof by Contradiction I

- Suppose we want to prove that p is true.
- Furthermore, suppose that we can find a contradiction q such that

$$\neg p \rightarrow q$$

is true.

- Because q is false, but $\neg p \rightarrow q$ is true, we can conclude that $\neg p$ is false, which means that p is true.

Proof by Contradiction II

- The statement

$$(r \wedge \neg r)$$

is a contradiction whenever r is a proposition.

- Therefore, we can prove that p is true if we can show that

$$\neg p \rightarrow (r \wedge \neg r)$$

is true for some proposition r .

Proof by Contradiction III

Example 9: Show that at least four of any 22 days must fall on the same day of the week.

Proof by Contradiction IV

- p : At least four of 22 days chosen fall on the same day of the week.
- Suppose $\neg p$ is true.
- This means that at most three days of the 22 days fall on the same day.
- This implies that at most 21 days could have been chosen.
- This contradicts the hypothesis that we have 22 days under consideration.
- r : 22 days are chosen.

$$\neg p \rightarrow (r \wedge \neg r)$$

$\sqrt{2}$ is irrational

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Reading Assignment

- Examples 11 – 14
- Subsection “Mistakes in Proofs”

Exhaustive Proof

Prove that $(n + 1)^2 \geq 3^n$ if n is a positive integer with $n \leq 4$.

Proof by Cases

Based on the logical equivalence

$$(p_1 \vee p_2 \vee \dots \vee p_n) \rightarrow q \equiv (p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \dots \wedge (p_n \rightarrow q)$$

Proof by Cases

Example 3: Prove that if n is an integer, then $n^2 \geq n$.

Proof by Cases

Example 4: Use a proof by cases to show that $|xy| = |x||y|$, where x and y are real numbers.

Proof by Cases

Example 5: Formulate a conjecture about the decimal digits that occur as the final digit of the square of an integer and prove your result.

Existence Proofs

- Many theorems are assertions that objects of a particular type exist.
- A theorem of this type is a proposition of the form $\exists xP(x)$, where P is a predicate.
- A proof of a proposition of the form $\exists xP(x)$ is called **existence proof**.

Constructive vs. Nonconstructive

- Constructive Existence Proof:

- Sometimes an existence proof can be given by finding a concrete element a such that $P(a)$ is T.

- Nonconstructive Existence Proof:

- We do not find an element a such that $P(a)$ is T, but rather prove that $\exists xP(x)$ is true in some other way.
- One common method of give a nonconstructive proof is to use proof by contradiction and show the negation of the existential quantifier implies a contradiction.

A Constructive Existence Proof

- Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

A Constructive Existence Proof

- Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.
- Solution: After considerable computation (such as computer search) we find that

$$1729 = 10^3 + 9^3 = 12^3 + 1^3 .$$

Because we have displayed a positive integer that can be written as the sum of cubes in two different ways, we are done.

Nonconstructive Existence Proof

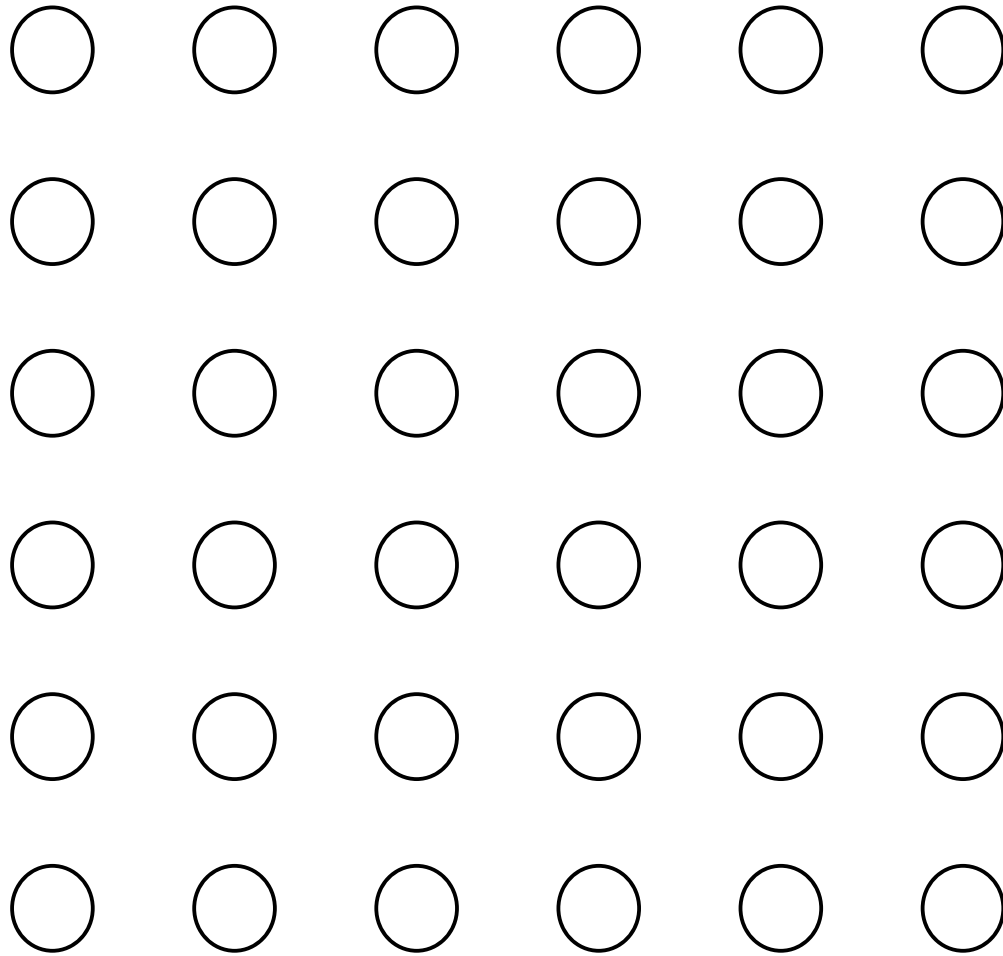
- Show that there exist irrational numbers x and y such that x^y is rational.

Nonconstructive Existence Proof

- Show that there exist irrational numbers x and y such that x^y is rational.
- Recall that $\sqrt{2}$ is irrational. Consider the number $\sqrt{2}^{\sqrt{2}}$.
 - If $\sqrt{2}^{\sqrt{2}}$ is rational, we have two irrational numbers x and y with x^y rational, namely, $x = \sqrt{2}$ and $y = \sqrt{2}$.
 - On the other hand if $\sqrt{2}^{\sqrt{2}}$ is irrational, then we can let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$ so that

$$x^y = \left(\sqrt{2}^{\sqrt{2}} \right)^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2} \cdot \sqrt{2})} = \sqrt{2}^2 = 2.$$

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Uniqueness Proof

- Some theorems assert the existence of a unique element with a particular property.
- The two parts of **uniqueness proof** are:
 - Existence: We show that there is an element x with the desired property.
 - Uniqueness: We show that if $y \neq x$, then y does not have the desired property.
- This is the same as showing that

$$\exists x(P(x) \wedge \forall y(y \neq x \rightarrow \neg P(y)))$$

is true.

Uniqueness Proof

Example 13: Show that if a and b are real numbers and $a \neq 0$, then there exists a unique real number r such that $ar + b = 0$.

Reading Assignment + Lab

- Read pages 93 – end of Chapter 1 on your own.
- The teaching assistants will go over material relevant to proofs during the lab sections.
- Proof strategies etc. are relevant for homework 2 (to be posted soon).

















