Intro to Discrete Structures
Lecture 6

Pawel M. Wocjan

School of Electrical Engineering and Computer Science

University of Central Florida

wocjan@eecs.ucf.edu

Hints for HW1

Hints for HW1

Formal Notation & Meaning

- Let $p_1, p_2, \ldots, p_n$ and c be (compound) propositions. The notation

$$\begin{array}{rl} & p_1 \\ & p_2 \\ & \vdots \\ & p_n \\ \hline \therefore & c \end{array}$$

means

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow c \equiv \mathsf{T}.$$

- $p_1, p_2, \ldots, p_n$ are called the premises and c is called the conclusion.

I. Rules of Inference

- Modus ponens

$$\begin{array}{rl} & p \\ & p \to q \\ \hline \therefore & q \end{array}$$

- Modus tollens

$$\begin{array}{rl} & \neg q \\ & p \to q \\ \hline \therefore & \neg p \end{array}$$

Modus Ponens – Modus Tollens

II. Rules of Inference

- Hypothetical syllogism

$$\begin{array}{rl} & p \rightarrow q \\ & q \rightarrow r \\ \hline \therefore & p \rightarrow r \end{array}$$

- Disjuctive syllogism

$$\begin{array}{rl} & p \vee q \\ & \neg p \\ \hline \therefore & q \end{array}$$

Modus Ponens – Disjunctive syllogism

III. Rules of Inference

- Addition

$$\begin{array}{ll} & p \\ \hline \therefore & p \vee q \end{array}$$

- Simplification

$$\begin{array}{ll} & p \wedge q \\ \hline \therefore & p \end{array}$$

IV. Rules of Inference

- Conjunction

$$\begin{array}{rl} & p \\ & q \\ \hline \therefore & p \wedge q \end{array}$$

- Resolution

$$\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$$

Resolution

- The inference rule "resolution"

$$\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$$

  is based on the tautology

$$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)\,.$$

- We recover disjunctive syllogism by setting $r = F$

$$((p \vee q) \wedge (\neg p)) \rightarrow q\,.$$

Hypothetical Syllogism – Resolution

Hypothetical Syllogism – Resolution

Rules of Inference for Quantified Statements

- Universal instantiation

$$\begin{array}{rl} & \forall x P(x) \\ \hline \therefore & P(c) \end{array}$$

- Universal generalization

$$\begin{array}{rl} & P(c) \text{ for an arbitrary } c \\ \hline \therefore & \forall x P(x) \end{array}$$

Rules of Inference for Quantified Statements

- Existential instantiation

$$\frac{\exists x P(x)}{\therefore \quad P(c) \text{ for some element } c}$$

- Existential generalization

$$\frac{P(c) \text{ for an some element } c}{\therefore \quad \exists x P(x)}$$

Universal Modus Ponens

- Universal Modus Ponens

$$
\begin{array}{rl}
 & \forall x(P(x) \rightarrow Q(x)) \\
 & P(a)\,, \text{ where } a \text{ is a particular element in the domain} \\
\hline
\therefore & Q(a)
\end{array}
$$

UMP in Math

- Assume that the statement

  "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n"

  is true. Use UMP to show that $100^2 < 2^{100}$.

Universal Modus Tollens

- Universal Modus Tollens

$$\begin{array}{rl} & \forall x(P(x) \rightarrow Q(x)) \\ & \neg Q(a)\,, \text{ where } a \text{ is a particular element in the domain} \\ \hline \therefore & \neg P(a) \end{array}$$

Introduction to Proofs

- Some Terminology
  - Axiom (or Postulate)
  - Theorem
  - Proposition
  - Lemma (Lemmas or Lemmata)
  - Corollary
  - Proof
  - Conjecture

Proofs Methods

- Direct Proof
- Proof by Contraposition (or Indirect Proof)
- Proof by Contradition
- Proof by Induction

Direct Proof

Definition 1: The integer n is **even** if there exists an integer k such that $n = 2k$, and n is **odd** if there exists an integer k such that $n = 2k + 1$.

Example 1: Give a direct proof of the theorem "If n is odd, then n^2 is odd."

Direct Proof

Definition: An integer a is a perfect square if there is an integer b such that $a = b^2$.

Example 2: Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square.

Indirect Proof

Example 3: Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

The direct approach does not work!

Indirect Proof

Example 3: Prove that if n is an integer and $3n+2$ is odd, then n is odd.

Use contraposition!

$$\forall n \in \mathbb{N}\, \big(\mathrm{odd}(3n+2) \rightarrow \mathrm{odd}(n)\big)$$

Indirect Proof

Example 4: Prove that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ and $b \leq \sqrt{n}$.

Proof Strategy

- We have seen two important methods for proving theorems of the form $\forall x(P(x) \rightarrow Q(x))$.

  These two methods are

  - the direct proof and
  - the indirect proof

  methods.

- It takes some practice (solving homework problems) to learn to recognize quickly the correct approach.
- Try first the direct approach. If it does not work then try the indirect approach.

Direct or Indirect Proof?

Definition: The real number r is **rational** if there exist integers p and q with $q \neq 0$ such that $r = p/q$.

Example 7: Prove that the sum of two rational numbers is rational.

Direct or Indirect Proof?

Example 8: Prove that if n is an integer and n^2 is odd, then n is odd.

Proof by Contradition

- We can prove that p is true if we can show that $\neg p \rightarrow (r \vee \neg r)$ is true for some proposition r.