Intro to Discrete Structures Lecture 6

Pawel M. Wocjan

School of Electrical Engineering and Computer Science
University of Central Florida

wocjan@eecs.ucf.edu

Hints for HW1

$$\frac{(p \rightarrow q) \land (q \rightarrow r))}{(p \rightarrow q)} \Rightarrow q \qquad \text{we} \qquad p \rightarrow q \equiv \neg p \lor q}$$

$$\frac{q \rightarrow r}{i \cdot p \rightarrow r} \qquad poposition 1 \qquad LAW$$

$$LAW \equiv poposition 2$$

$$\equiv \frac{1}{2} = \frac{1}{2}$$

Hints for HW1

Formal Notation & Meaning

• Let p_1, p_2, \ldots, p_n and c be (compound) propositions. The notation

$$p_1$$
 p_2
 \vdots
 p_n

means

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow c \equiv \mathsf{T}.$$

• p_1, p_2, \ldots, p_n are called the premises and c is called the conclusion.

I. Rules of Inference

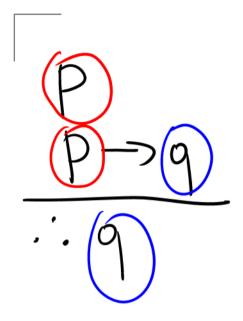
Modus ponens

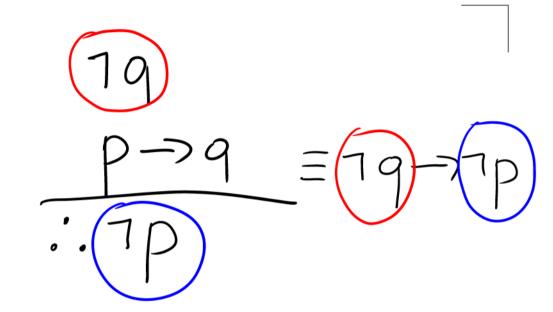
$$\begin{array}{c}
p \\
p \to q \\
\hline
\vdots \quad q
\end{array}$$

Modus tollens

$$\begin{array}{c}
\neg q \\
p \to q \\
\hline
\vdots \quad \neg p
\end{array}$$

Modus Ponens – Modus Tollens





II. Rules of Inference

Hypothetical syllogism

$$\begin{array}{c}
p \to q \\
q \to r \\
\hline
\vdots \quad p \to r
\end{array}$$

Disjuctive syllogism

$$\begin{array}{c}
p \lor q \\
\hline
\neg p \\
\hline
\vdots \quad q
\end{array}$$

Modus Ponens – Disjunctive syllogism

III. Rules of Inference

Addition

$$\begin{array}{c} p \\ \vdots \quad p \vee q \end{array}$$

Simplification

$$\frac{p \wedge q}{\therefore p}$$

IV. Rules of Inference

Conjunction

$$\begin{array}{c}
p \\
q \\
\hline
 \therefore p \land q
\end{array}$$

 $(p \wedge q) \rightarrow (p \wedge q) = T$

Resolution

$$\begin{array}{c}
p \lor q \\
\neg p \lor r \\
\hline
\vdots \quad q \lor r
\end{array}$$

Resolution

The inference rule "resolution"

$$\begin{array}{c}
p \lor q \\
\neg p \lor r \\
\hline
\therefore q \lor r
\end{array}$$

is based on the tautology

$$((p \lor q) \land (\neg p \lor r)) \to (q \lor r).$$

• We recover disjunctive syllogism by setting r = F

$$((p \lor q) \land (\neg p)) \rightarrow q$$
.

Hypothetical Syllogism – Resolution

$$PV9 = 7977$$

$$7PVY = P77$$

$$9VY = 797$$

Hypothetical Syllogism – Resolution

Rules of Inference for Quantified Statements

Universal instantiation

Universal generalization

$$P(c)$$
 for an arbitrary c
 $\therefore \forall x P(x)$

Rules of Inference for Quantified Statements

Existential instantiation

$$\exists x P(x)$$
 $\therefore P(c) \text{ for some element } c$

Existential generalization

$$P(c)$$
 for an some element c
 $\therefore \exists x P(x)$

Universal Modus Ponens

Universal Modus Ponens

$$\forall x(P(x) \rightarrow Q(x))$$
 $P(a)$, where a is a particular element in the domain
 $\therefore Q(a)$

All Greeks are Mortal
$$\forall x (6(x) \rightarrow M(x))$$

Solvales is a Greek $6(Sokrales)$
Therefore, Solvales is ... $M(Sokrales)$
Mortal.

UMP in Math

Assume that the statement

"For all positive integers n, if n is greater than 4, then n^2 is less than 2^n "

is true. Use UMP to show that $100^2 < 2^{100}$.

$$P(n): n > 4$$

 $Q(n): n^{2} < 2^{n}$
 $\forall n (P(n) \rightarrow Q(n))$
 $P(100)$
 $Q(100)$

Universal Modus Tollens

Universal Modus Tollens

$$\forall x(P(x) \rightarrow Q(x))$$

 $\neg Q(a)$, where a is a particular element in the domain
 $\therefore \neg P(a)$

Introduction to Proofs

- Some Terminology
 - Axiom (or Postulate)
 - Theorem
 - Proposition
 - Lemma (Lemmas or Lemmata)
 - Corollary
 - Proof
 - Conjecture

Proofs Methods

- Direct Proof
- Proof by Contraposition (or Indirect Proof)
- Proof by Contradition
- Proof by Induction

Direct Proof

Definition 1: The integer n is **even** if there exists an integer k such that n=2k, and n is **odd** if there exists an integer k such that n=2k+1.

Example 1: Give a direct proof of the theorem "If n is odd, then n^2 is odd."

Proof: assume
$$\pi$$
 is odd =>= $\frac{1}{2}k \pi = 2k+1$
=>= $\frac{1}{2}k \pi^{2} = (2k+1)^{2} = (2k)^{2} + 2(2k) + 1^{2}$
== $\frac{1}{2}k^{2} + 4k + 1$
== $\frac{1}{2}(2k^{2} + 2k) + 1$
== $\frac{1}{2}k^{2} + 2k$
=> $\frac{1}{2}$ is oold

Direct Proof

Definition: An integer a is a perfect square if there is an integer b such that $a=b^2$.

Example 2: Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square.

$$m = r^2$$
 $n = s^2$
 $m = r^2 \cdot r_1 s$ are integers
$$m = r^2 \cdot s^2 = (rs)^{2i} = t^2 \text{ where}$$

$$t = rs$$

Indirect Proof

Example 3: Prove that if n is an integer and 3n + 2 is odd, then n is odd.

The direct approach does not work!

$$3\pi + 2$$
 is odd $\Rightarrow 3\pi + 2 = 2k + 1$ for some integer k

Indirect Proof

Example 3: Prove that if n is an integer and 3n + 2 is odd, then n is odd.

Use contraposition!

$$= \forall n \in \mathbb{N} \left(\operatorname{odd}(3n+2) \to \operatorname{odd}(n) \right)$$

$$= \forall n \in \mathbb{N} \left(\operatorname{rodd}(n) \longrightarrow \operatorname{rodd}(3n+2) \right)$$

$$= \forall n \in \mathbb{N} \left(\operatorname{even}(n) \longrightarrow \operatorname{even}(3n+2) \right)$$

$$= 2k \quad \text{for some integer } k$$

$$= 3n+2 = 3(2k)+2 = 6k+421$$

$$= 2(3k+1) = 3k+1$$

$$= 2(3k+1) = 3k+1$$

Indirect Proof

Example 4: Prove that if n=ab, where a and b are positive integers, then $a \leq \sqrt{n}$ and $b \leq \sqrt{n}$.

Proof Strategy

• We have seen two important methods for proving theorems of the form $\forall x (P(x) \rightarrow Q(x))$.

These two methods are

- the direct proof and
- the indirect proof

methods.

- It takes some practice (solving homework problems) to learn to recognize quickly the correct approach.
- Try first the direct approach. If it does not work then try the indirect approach.
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Direct or Indirect Proof?

Definition: The real number r is **rational** if there exist integers p and q with $q \neq 0$ such that r = p/q.

Example 7: Prove that the sum of two rational numbers is rational.

Direct or Indirect Proof?

Example 8: Prove that if n is an integer and n^2 is odd, then n is odd.

Proof by Contradition

• We can prove that p is true if we can show that $\neg p \rightarrow (r \lor \neg r)$ is true for some proposition r.