Intro to Discrete Structures
Lecture 5

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Office Hours

- Tuesday 2:45pm – 4:00pm
- Thursday 1:00pm – 2:15pm

Nested Quantifiers

- Two quantifiers are **nested** if one is within the scope of the other, such as

$$\forall x \exists y (x + y = 0) \, .$$

- Everything within the scope of a quantifier can be thought of as a propositional function. Define the propositional functions
  - $Q(x) \ : \ \exists y P(x, y)$
  - $P(x, y) \ : \ x + y = 0$
- Then, we have

$$\forall x \exists y (x + y = 0) \equiv \forall x \exists y P(x, y) \equiv \forall x Q(x) \, .$$

Thinking of Quantification as Loops

- In working with quantification of more than one variable, it is sometimes helpful to think in terms of nested loops.

  Of course, if there are infinitely many elements in the domain of some variable, we cannot actually loop through all values. Nevertheless, this way of thinking is helpful in understanding nested quantifiers.

- For example, $\forall x \forall y P(x, y)$ corresponds to

```
for x do
  for y do
    if ¬P(x,y) return F
  end for
end for
return T
```

Thinking of Quantification as Loops

- For example, $\forall x \exists y P(x, y)$ corresponds to

```
for x do
temp:=F
  for y do
    if P(x,y) then
      temp:=T;
      break
  end for
  if ¬temp return F
end for
return T
```

Order of Quantifiers

- $\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$
- True if $P(x, y)$ is T for every pair x, y.
- False if there is a pair x, y for which $P(x, y)$ is F.

- $\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$
- True if there is a pair x, y for which $P(x, y)$ is T.
- False if $P(x, y)$ is F for every pair x, y.

Order of Quantifiers

- $\forall x \exists y P(x, y)$
- True if for every x there is a y for which $P(x, y)$ is T.
- False if there is an x such that $P(x, y)$ is F for every y.

- $\exists x \forall y P(x, y)$
- True if there is an x for which $P(x, y)$ is T for every y.
- False if for every x there is a y for which $P(x, y)$ is F.

Reading Assignment

- Translating Mathematical Statements into Statements Involving Nested Quantifiers
- Translating from Nested Quantifiers to English
- Translating English Sentences into Logical Expressions

Negating Nested Quantifiers

Example 14: Express the negation of $\forall x \exists y (xy = 1)$ so that no negation precedes a quantifier.

$$
\begin{aligned}
& & & \neg \forall x \exists y (xy = 1) \\
\text{De Morgan} & & \equiv\ & \exists x \neg \exists y (xy = 1) \\
\text{De Morgan} & & \equiv\ & \exists x \forall y \neg (xy = 1) \\
& & \equiv\ & \exists x \forall y (xy \neq 1)
\end{aligned}
$$

Truth Value of Quantified Expressions

- Determine the truth value of $\forall x \exists y (xy = 1)$. Assume that the domain is equal to $\mathbb{R}$.

Truth Value of Quantified Expressions

- Determine the truth value of $\forall x \exists y (xy = 1)$. Assume that the domain is equal to $\mathbb{R}$.

- $\forall x \exists y (xy = 1) \equiv F$ because for $x = 0$ there is no y such that $0 \cdot y = 1$.

- This statement becomes T if we change the domain to be equal to $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ (all real numbers except for 0).

  In words, this statement means that every nonzero real x number has a multiplicative inverse $y = 1/x$.

Rules of Inference

- Proof in mathematics are valid arguments that establish the truth of mathematical statements.

- By **argument**, we mean a sequence of statements that end with a conclusion.

  By **valid**, we mean that the conclusion, or final statement of the argument, must follow logically from the truth of the preceeding statements, or **premises**, of the argument.

- That is, an argument is valid if and only if it is impossible for all the premises to be true and the conclusion to be false.

Example

- "If you have current password, then you can log onto the network."
- "You have a current password."

Therefore,

- "You can log onto the network."

Formal notation

- Formally, we write

$$\begin{array}{rl} & p \rightarrow q \\ & p \\ \hline \therefore & q \end{array}$$

Formal Notation & Meaning

- Let $p_1, p_2, \ldots, p_n$ and c be (compound) propositions. The notation

$$\begin{array}{rl} & p_1 \\ & p_2 \\ & \vdots \\ & p_n \\ \hline \therefore & c \end{array}$$

means

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow c \equiv \mathsf{T}.$$

- $p_1, p_2, \ldots, p_n$ are called the premises and c is called the conclusion.

I. Rules of Inference

- Modus ponens

$$\begin{array}{rl} & p \\ & p \rightarrow q \\ \hline \therefore & q \end{array}$$

- Modus tollens

$$\begin{array}{rl} & \neg q \\ & p \rightarrow q \\ \hline \therefore & \neg p \end{array}$$

II. Rules of Inference

- Hypothetical syllogism

$$\begin{array}{rr} & p \rightarrow q \\ & q \rightarrow r \\ \hline \therefore & p \rightarrow r \end{array}$$

- Disjuctive syllogism

$$\begin{array}{rr} & p \vee q \\ & \neg p \\ \hline \therefore & q \end{array}$$

III. Rules of Inference

- Addition

$$\begin{array}{ll} & p \\ \hline \therefore & p \vee q \end{array}$$

- Simplification

$$\begin{array}{ll} & p \wedge q \\ \hline \therefore & p \end{array}$$

IV. Rules of Inference

- Conjunction

$$\begin{array}{rl} & p \\ & q \\ \hline \therefore & p \wedge q \end{array}$$

- Resolution

$$\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$$

Resolution

- The inference rule "resolution"

$$\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$$

  is based on the tautology

$$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)\,.$$

- We recover disjunctive syllogism by setting $r = F$

$$((p \vee q) \wedge (\neg p)) \to q\,.$$

Rules of Inference for Quantified Statements

- Universal instantiation

$$\frac{\forall x P(x)}{\therefore \quad P(c)}$$

- Universal generalization

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \quad \forall x P(x)}$$

Rules of Inference for Quantified Statements

- Existential instantiation

$$\frac{\exists x P(x)}{\therefore \quad P(c) \text{ for some element } c}$$

- Existential generalization

$$\frac{P(c) \text{ for an some element } c}{\therefore \quad \exists x P(x)}$$

Universal Modus Ponens

- Universal Modus Ponens

$$\begin{array}{rl} & \forall x(P(x) \to Q(x)) \\ & P(a)\,, \text{ where } a \text{ is a particular element in the domain} \\ \hline \therefore & Q(a) \end{array}$$

- Assume that (*) "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n" is true. Use UMP to show that $100^2 < 2^{100}$.
  - Let $P(n) : n > 4$ and $Q(n) : n^2 < 2^n$.
  - The statement (*) corresponds to $\forall n(P(n) \to Q(n))$.
  - UMT implies that $Q(100)$ is T.
  - $P(100)$ is T.

Universal Modus Tollens

- Universal Modus Tollens

$$
\begin{array}{rl}
 & \forall x (P(x) \rightarrow Q(x)) \\
 & \neg Q(a)\,,\ \text{where } a \text{ is a particular element in the domain} \\
\hline
\therefore & \neg P(a)
\end{array}
$$

Introduction to Proofs

- Some Terminology
  - Axioms (or postulates)
  - Theorem
  - Proposition
  - Lemma (Lemmas or Lemmata)
  - Corollary
  - Proof
  - Conjecture