Intro to Discrete Structures
Lecture 4

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Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have a higher precedence then all logical operators from propositional calculus.
- For example,

$$\forall x\, P(x) \vee Q(x)$$

means

$$\big(\forall x\, P(x)\big) \vee Q(x)$$

rather than

$$\forall x\, \big(P(x) \vee Q(x)\big)\, .$$

Binding Variables

- When a quantifier is used on the variable x, we say that this occurence of the variable is **bound**.
- An occurence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**.
- All variables that occur in a predicate must be bound or set equal to a particular value to turn it into a proposition.

  This can be done by a combination of $\forall$, $\exists$, and value assignments.
- The part of a logical expression to which a quantifier is applied is called the **scope** of this quantifier.

Binding Variables – Examples

- $$\exists x(x + y = 1)$$
  - The variable x is bound by the existential quantifier.
  - The variable y is free because it is not bound by a quantifier and no value is assigned to this variable.

Binding Variables – Examples

- $$\exists x \big(P(x) \land Q(x)\big) \lor \forall x R(x)$$
  - All variables are bound.
  - The scope of the first quantifier, $\exists x$, is the expression $P(x) \land Q(x)$.
  - The scope of the second quantifier, $\forall x$, is the expression $R(x)$.

    We could have written the above expression using two different variables x and y as

    $$\exists x \big(P(x) \land Q(x)\big) \lor \forall y R(y)$$

    because the scopes do not overlap.

Logical Equivalence

- Definition 3: Expressions involving predicates and quantifiers are called **logically equivalent** if and only if they have the same truth value no matter
  - which predicates are substituted into these statements and
  - which domain of discourse is used for the variables in these variables in these predicates.
- We use the notation $S \equiv T$ that the two expressions S and T involving predicates and quantifiers are logically equivalent.

Logical Equivalence - Example

- Prove

$$\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)\,.$$

- To this end, we show:
  - if $\forall x(P(x) \wedge Q(x))$ is T, then $\forall x P(x) \wedge \forall x Q(x)$ is T
  - if $\forall x P(x) \wedge \forall x Q(x)$ is T, then $\forall x(P(x) \wedge Q(x))$ is T

Logical Equivalence – Example

- So, suppose that $\forall x(P(x) \wedge Q(x))$ is T.
- This means that if the element a is in the domain, then $P(a) \wedge Q(a)$ is T.
- Hence, $P(a)$ is T and $Q(a)$ is T.
- Because $P(a)$ and $Q(a)$ is T for every element a in the domain, we can conclude that $\forall x P(x)$ and $\forall x Q(x)$ are both T.
- This means that $\forall x(P(x) \wedge Q(x))$ is T.

Logical Equivalence – Example

- Next, suppose that $\forall x P(x) \wedge \forall x Q(x)$ is T.
- This means that $\forall x P(x)$ is T and $\forall x Q(x)$ is T.
- Hence, if a and b are two arbitrary elements in the domain, then $P(a)$ and $Q(b)$ are T.
- Hence, if a is an arbitrary element in the domain $P(a)$ and $Q(a)$ are T (we may choose b to be equal to a).
- It follows that $\forall x (P(x) \wedge Q(x))$ is T.

Logical Equivalences and Nonequivalences

$$\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$$

$$\exists x(P(x) \lor Q(x)) \equiv \exists x P(x) \lor \exists x Q(x)$$

$$\forall x(P(x) \lor Q(x)) \not\equiv \forall x P(x) \lor \forall x Q(x)$$

$$\exists x(P(x) \land Q(x)) \not\equiv \exists x P(x) \land \exists x Q(x)$$

Negating Quantified Expressions

- We often want to consider the negation of a quantified expression.
- For instance, consider the statement

  "Every student enrolled in COT3100 will get an F."

- Its negation is

  "There is at least one student enrolled in COT3100 who will get a grade better than F for this course."

Negation of Existential Quantifier

- Negation of existential quantifier

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

  - Negation is T if for every x, $P(x)$ is F.
  - Negation is F if there exists an x for which $P(x)$ is T.

Negation of Universal Quantifier

- Negation of universal quantifier

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

  - Negation is T if there is an x for which $P(x)$ is F.
  - Negation is F if $P(x)$ is T for every x.

De Morgan's Laws for Quantifiers

- De Morgan's Laws are

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

- Remark: When the domain of a predicate $P(x)$ consists of m elements, where m is a positive integer, then the above rules of negating quantified predicates are exactly the same as De Morgan's Laws in Propositional Logic.

Logical Equivalence

Example 22: Show that

$$\neg\forall x(P(x) \rightarrow Q(x)) \equiv \exists x(P(x) \wedge \neg Q(x))\,.$$

$$\begin{aligned}
& \neg\forall x(P(x) \rightarrow Q(x)) \\
\text{De Morgan} \quad \equiv \quad & \exists x(\neg(P(x) \rightarrow Q(x))) \\
\text{impl-or} \quad \equiv \quad & \exists x(\neg(\neg P(x) \vee Q(x))) \\
\text{De Morgan} \quad \equiv \quad & \exists x(\neg\neg P(x) \wedge \neg Q(x)) \\
\text{double negation} \quad \equiv \quad & \exists x(P(x) \wedge \neg Q(x)) \quad \square
\end{aligned}$$

Reading Assignment

- Translating from English into Logical Expressions
- Using Quantifiers in Systems Specifications
- Examples from Lewis Carroll
- Logic Programming