

Intro to Discrete Structures

Lecture 2

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Recap of Lecture 1

- propositions, propositional calculus (or propositional logic)
- propositional variables (or statement variables)
- truth value, T, F
- compound propositions, logical operators
- negation \neg
- connectives: conjunction \wedge , disjunction \vee , exclusive or \oplus , conditional statement \rightarrow , and biconditional statement \leftrightarrow
- hypothesis (or antecedent or premise) \rightarrow conclusion (or consequence)

Recap of Lecture 1

Enter the missing truth values into the truth table:

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T						
T	F						
F	T						
F	F						

Recap of Lecture 1

Enter the missing truth values into the truth table:

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

Translating English Sentences

Parent: If you don't clean your room, you can't watch a DVD.

$$\begin{array}{l} \neg C \quad \rightarrow \quad \neg D \quad \text{and} \\ C \quad \rightarrow \quad D \end{array}$$

means

$$C \quad \leftrightarrow \quad D$$

Implicit use of biconditionals: You should be aware that biconditional are not always explicitly used in natural language.

Translating English Sentences

Mathematician: If a function is not **continuous**, then it is not **differentiable**.

$$\neg C \rightarrow \neg D$$

but

$$C \rightarrow D$$

is not implied!

For instance, the absolute value function $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto |x|$ is continuous, but it is not differentiable since it doesn't have a well defined tangent at $x = 0$.

System Specification

Boolean Searches

Logic and Bit Operations

Logic Puzzle – Alien Encounter

The starship Indefensible is in orbit around the planet Noncomposmentis, and Captain Quirk and Mr Crock have just beamed down to the surface.

Alien Encounter

- Quirk: 'According to the *Good Galaxy Guide*, there are two species of intelligent aliens on this planet.'
- Crook: 'Correct, Captain - Veracitors and Gibberish. They all speak Galaxix, and they can be distinguished by how they answer questions. The Veracitors always reply truthfully, and the Gibberish always lie.'
- Quirk: 'But physically –'
- Crook: 'They are indistinguishable, Captain.'

Alien Encounter

Quirk hears a sound, and turns to find three aliens creeping up on them. They look identical.

Alien Encounter

- One of the Aliens: 'Welcome to Noncomposmentis.'
- Quirk: 'I thank you. My name is Quirk. Now, you are ...'
Quirk pauses. 'No point in asking their names,' he mutters. 'For all we know, they'll be wrong.'
- Crook: 'That is logical, Captain.'
- Quirk: 'Because we are poor speakers of Galaxic, I hope you will not mind if I call you Alf, Betty and Gemma.' As he speaks, he points to each of them in turn. Then he turns to Crook and whispers, 'Not that we know what sex they are, either.'
- Crook: 'They are all hermandrofemigynes.'

Alien Encounter

- Quirk: 'Whatever. Now, Alf, to which species does Betty belong?'
- Alf: 'Gibberish.'
- Quirk: 'Ah. Betty, do Alf and Gemma belong to different species?'
- Betty: 'No.'
- Quirk: 'Right ... Talkative lot, aren't they? Um ... Gemma, to which species does Betty belong?'
- Gemma: 'Veracitor.'

Alien Encounter

- Quirk: 'Right, that's settled it, then!' He nods knowledgeably.
- Crook: 'Settled what, Captain?'
- Quirk: 'Which species each belongs to.'

Alien Encounter

Are you as smart as Captain Quirk? Do you know which species each belongs to?

I. Translating Galaxix to Logix

• α says

$$\beta = G$$

• β says

$$\neg(\alpha \neq \gamma) \Leftrightarrow (\alpha = \gamma)$$

• γ says

$$\beta = V$$

• assume $\alpha = V$ (first possibility)

$$\Rightarrow \beta = G \Rightarrow \alpha \neq \gamma \Rightarrow \gamma = G \Rightarrow \beta \neq V \Rightarrow \beta = G$$

II. Translating Galaxix to Logix

• α says

$$\beta = G$$

• β says

$$\neg(\alpha \neq \gamma) \Leftrightarrow (\alpha = \gamma)$$

• γ says

$$\beta = V$$

• assume $\alpha = G$ (second possibility)

$$\Rightarrow \beta = V \Rightarrow \gamma = G \Rightarrow \beta \neq V \text{ contradiction}$$

Extending the Puzzle???

Can we always ask the right questions?

That is, can we always formulate questions that allow us to deduce the three aliens' correct specie types, no matter what they are?

There are 8 different configurations that are possible
 $(\alpha, \beta, \gamma) \in \{VVV, VVG, \dots, GGG\}$.

1.2. Propositional Equivalence

Definition 1: A compound proposition that is always true, no matter what the truth value of the propositions that occur in it, is called a **tautology**.

A compound proposition that is always false is called a **contradiction**.

A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

Tautologies and contradictions are often important in mathematical reasoning.

Example of a Tautology and a Contradiction

We can construct examples of tautologies and contradictions using just one proposition.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logical Equivalence

Compound propositions that have the same truth value in all possible cases are called **logically equivalent**.

We can also define this notion as follows.

Definition 2: The compound propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Remark: The symbol \equiv is not a logical connective and $p \equiv q$ is not a compound proposition, but rather is the statement that $p \leftrightarrow q$ is a tautology. The symbol \Leftrightarrow is sometimes used instead of \equiv .

Logical Equivalence

One way to determine whether two compound propositions are equivalent is to use a truth table. In particular, the compound propositions p and q are equivalent if and only if the columns giving their truth values agree.

Example 2: Show that $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

It follows that $\neg(p \vee q) \equiv \neg p \wedge \neg q$ holds, which is the first of the two DeMorgan Laws.

De Morgan Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Logical Equivalences

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg\neg p \equiv p$	Double negation
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws

Further laws are the associative, distributive, De Morgan's, absorption, negation laws (see page 24).

Further Logical Equivalences

See page 25 for logical equivalences involving conditional and biconditional statements.

Using De Morgan's Laws

Constructing New Logical Equivalences

Which of the following two compound propositions

$$(G \rightarrow S) \vee (G \rightarrow J)$$

$$G \rightarrow (S \vee J)$$

is the correct translation of

Ελληνες κρατανε σπαθια η ακοντια.

into propositional logic?

Which of the following two compound propositions

$$(G \rightarrow S) \vee (G \rightarrow J)$$

$$G \rightarrow (S \vee J)$$

is the correct translation of “Greeks carry Swords or Javelins”

Ελληνες κρατανε σπαθια η ακοντια.

into propositional logic?

It turns out that both compound propositions are equivalent.

How do we show that? One approach is via building the truth table and comparing the corresponding columns.

Let's do something fancier.

First, convince yourself first that the following laws are correct:

$p \rightarrow q$	\equiv	$\neg p \vee q$	implication in terms of or	(impl-or)
$p \vee q$	\equiv	$q \vee p$	commutative law	(comm)
$p \vee p$	\equiv	p	idempotent law	(idem)
$p \vee (q \vee r)$	\equiv	$(p \vee q) \vee r$	associative law	(ass)

300

Second, apply these laws in a 'smart' way:

$$(G \rightarrow S) \vee (G \rightarrow J)$$

apply the law

$$\text{impl-or} \quad \equiv \quad (\neg G \vee S) \vee (\neg G \vee J)$$

$$\text{ass} \quad \equiv \quad \neg G \vee S \vee \neg G \vee J$$

$$\text{comm} \quad \equiv \quad \neg G \vee \neg G \vee S \vee J$$

$$\text{idem} \quad \equiv \quad \neg G \vee S \vee J$$

$$\text{ass} \quad \equiv \quad \neg G \vee (S \vee J)$$

$$\text{impl-or} \quad \equiv \quad G \rightarrow (S \vee J) \quad \square$$

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Ultimate conclusion: watching trashy movies like 300 is not a waste of time?

1.3. Predicates and Quantifiers

A declarative sentence is an **predicate** if

- it contains one or more variables, and
- it is not a proposition, but
- it becomes a proposition when the variables in it are replaced by certain by certain allowable choices.

The allowable choices constitute what is called the **universe** (or **universe of discourse**) for the predicate.

Predicates

- When we examine the sentence “The number $x + 2$ is greater than 1” in light of this definition, we find that it is a predicate that contains the single variable x .
- The universe could be the natural numbers \mathbb{N} , the integers \mathbb{Z} , the rational numbers \mathbb{Q} , or the real numbers \mathbb{R} .
- We choose the universe to be \mathbb{R} .
- Let us use $P(x)$ as a short hand notation for “The number $x + 2$ is strictly greater than 1.”
- Determine the truth values of $P(-\sqrt{2})$ and $P(-0.5)$.

Predicates

- Recall that $P(x)$ denotes the predicate “The number $x + 2$ is strictly greater than 1.”
- We have $-\sqrt{2} + 2 \leq 1$, so the truth value of $P(-\sqrt{2})$ is F.
- We have $-0.5 + 2 = 1.5 > 1$, so the truth value of $P(-0.5)$ is T.