Problem 1.

(a) Place each of the consonants out with gaps in between them:

_ M _ S _ S _ S _ S _ P _ P _

We can choose 4 of these 8 locations to place the Is in ${}_8C_4$ ways. **(3 pts)**

Independently, we can permute the consonants in $\frac{7!}{4!2!}$ ways. **(3 pts)** Thus, the total number of permutations is $\binom{8}{4}\frac{7!}{4!2!}$ to arrange the letters with the given restriction. **(1 pt).**

(b) For each digit, 1 through nine, we can either choose to place it in the number or not. Once these choices are made, the number if fixed. Thus, there are 2^9 possible choices. **(6 pts)**

But, one of those possibilities, not taking the digits for all 9 digits, results in an "empty" number. Thus, we don't count this one and have a total of $2^9 - 1$ ascending numbers. **(2 pts)**

Problem 2:

a. (1 point)

Given: $(x_1+x_2+\ldots+x_m)^n$

To find: number of terms with identical exponents in the expansion

(1 point)

Inference: This is an instance of Indistinguishable objects and Indistinguishable boxes.

Solution:

Consider an example, $(x_1+x_2)^2$

The expansion of this is $x_1^2 + x_2^2 + 2\, x_1\, x_2$.

Now terms with identical exponents are x_1^2 and x_2^2, and $2\, x_1\, x_2$.

When grouped together, x_1^2 and x_2^2 will belong to one group.

$2\, x_1\, x_2$ will belong to another group.

The number of elements in each group will give us the number of terms with identical exponents.

So, here we have

2 terms with the exponent 2

1 term with the exponent 1

The intuition is that the exponent has to be correctly distributed among the terms. If the equation consists of two elements and the exponent is two, like our example here, then the exponent will be shared by both elements, until it can no longer be shared as an integer. This means, the exponent 2 is shared by x_1 and x_2 alike with $x_1^2 + x_2^2 + 2 x_1 x_2$

Consider the following examples
$(a + b)^0 = 1$
$(a + b)^1 = a + b$
$(a + b)^2 = a^2 + 2ab + b^2$
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$
(Source: http://www.intmath.com/Series-binomial-theorem/4_Binomial-theorem.php)

Observe that the exponent is shared as an integer among the elements of the equation. This leads us to the answer for this question.

If there are m elements in an equation having an n exponent, then the number of elements having identical exponents will be the number of ways n can be expressed as a sum of positive integers shared by the elements i.e. C(n+m-1, m-1).

Grading:
4 points if the logic is correct.
2 points if the problem is attempted.

2b.

Using the intuition of Binomial Coefficient theorem, x has to be chosen in two of ten terms, y has to be chosen in two of eight terms and consequently, z has to be chosen from remaining six terms. (3 points)

This leads to the product rule, (2 points)

C(10,2)C(8,2)C(6,6) = 45.28.1 = 1260 (1 point)

1 point for attempting the problem.

Problem 3.

3a. Given: A{1,2,3,4,5}, R {(1,1),(1,3),(1,5),(2,2),(2,4),(3,1),(3,3),(4,2),(5,1),(5,3),(5,5)}

Reflexive:
R is not reflexive, because (4,4) does not belong to R, but 4 belongs to A

Symmetric:
R is not symmetric, because (3,5) does not belong to R, but (5,3) belongs to R.

Anti-symmetric:

R is not anti-symmetric, because (1,3) and (3,1) belong to R, but 1 is not equal to 3

Transitive:

R is not transitive, because (3,1) and (1,5) belong to R, but (3,5) does not belong to R.

Grading: 2 points for each.

3b. The composition is not transitive, since the domain of S need not be the range of R. Consider a counter-example, R = {(1,1), (2,2), (9,3)} and S = {(1,1), (2,2), (3,3), (2,9)}, then R $\circ$ S = {(1,1), (2,2), (9,3), (2,9)} , which is not transitive.

Grading: 3 points for example, 2 points for reasoning, 1 point for attempting the problem

3c. Given: $(a,b)\in R$ iff $a^2+ b = 2k$, for some $k \in Z$, the set of integers.

To prove: R is an equivalence relation

Proof: A relation R is an equivalence relation if it is reflexive, symmetric, and transitive. (1 point)

Reflexive: $\forall a\in Z, (a,a) \in R$

Prove that $a^2+ a = 2k$, for some $k \in Z$, the set of integers.

Let a be an even number, 2c.

$a^2+ a = 4c^2+ 2c = 2(2c^2+ c)$, and let $(2c^2+ c) = k$. Therefore, $a^2+ a = 2k$.

Let a be an odd number, 2c+1.

$a^2+ a = 4c^2+ 4c + 1 + 2c + 1 = 4c^2+ 6c +2 = 2(2c^2+ 3c + 1)$, and let $(2c^2+ 3c + 1) = k$.

Therefore, $a^2+ a = 2k$

$\forall a\in Z, (a,a) \in R$ is therefore true. (1 Point)

Symmetric: $\forall a\forall b\in Z, (a,b) \in R \wedge (b,a) \in R$

Prove that $a^2+ b = 2k$ and $b^2+ a = 2m$ for some $k , m \in Z$, the set of integers.

Case 1: a and b are even

Let $a = 2c$ and $b = 2d$, then $a^2+ b = 4c^2+ 2d = 2(2c^2+ d)$. Let $(2c^2+ d) = k$. Therefore, $a^2+ b = 2k$.

Again, $b^2+ a = 4d^2+ 2c = 2(2d^2+ c)$. Let $(2d^2+ c) = m$. Therefore, $b^2+ a = 2m$. Proved.

Case 2: a and b are odd

Let $a = 2c + 1$ and $b = 2d + 1$, then $a^2+ b = 4c^2+4c+ 2+ 2d = 2(2c^2+ 2c + d+1)$. Let $(2c^2+ 2c+ d+1) = k$. Therefore, $a^2+ b = 2k$.

Again, $b^2+ a = 4d^2+4d+1+2c+1 = 2(2d^2+ 2d + d+1)$. Let $(2d^2+ 2d+ c+1) = k$. Therefore, $b^2+ a = 2m$. Proved.

Case 3: a is odd and b is even. Let $a = 1$, and $b = 2$, $a^2+ b = 1 + 2 = 3$, which is not an even number. Therefore, R does not contain entries with an odd a and an even b.

Case 4: a is even, b is odd. Let $a = 2$, and $b = 1$. $a^2+ b = 4 + 1 = 5$, which is not an even number. Therefore, R does not contain entries with an even a and an odd b. (2 Points)

Transitive relation: $\forall a \forall b \forall c \in Z, (a,b) \in R \wedge (b,c) \in R \Rightarrow (a,c) \in R$

Case 1: a, b, and c are all even

$a = 2d$, $b = 2m$, and $c = 2n$

$a^2+ b = 4d^2+2m = 2(2d^2+m)$, let $(2d^2+m) = k$, then $a^2+ b = 2k$ holds.

$b^2+ c = 4m^2+2n = 2(2m^2+n)$, let $(2m^2+n) = k$, then $b^2+ c = 2k$ holds.

$a^2+c = 4d^2+2n = 2(2d^2+n)$, let $(2d^2+n) = k$, then $a^2+ c = 2k$ holds.

Since they all satisfy the condition on the relation, transitivity is true for all even numbers in Z.

Case 1: a, b, and c are all odd

$a = 2d+1$, $b = 2m+1$, and $c = 2n+1$

$a^2+ b = 4d^2+4d+2+2m = 2(2d^2+m+2d+1)$, let $(2d^2+m+2d+1) = k$, then $a^2+ b = 2k$ holds.

$b^2+ c = 4m^2+4m+2+2n = 2(2m^2+n+2m+1)$, let $(2m^2+n+2m+1) = k$, then $b^2+ c = 2k$ holds.

$a^2+c = 2(2d^2+n+2d+1)$, let $(2d^2+n+2d+1) = k$, then $a^2+ c = 2k$ holds. (2 points)

Since they all satisfy the condition on the relation, transitivity is true for all odd numbers in Z.

From the proof of Symmetry, we know that there is no odd-even mixture in R. Thus, we have proved R to be an equivalence relation.

General Grading scheme:

- 1 point is automatically awarded is the problem is attempted.
- However, this will not apply, unless stated, if the problem has been correctly solved.