COT 3100 Discrete Mathematics
Homework 3 Key

March 2, 2010

- **Problem 1** Evaluate each of the following summations.

  (a) (2 points) $\sum_{k=1}^{5}(k+1)$

  $\sum_{k=1}^{5}(k+1)$
  $= (1+1)+(2+1)+(3+1)+(4+1)+(5+1)$
  $= 20$

  Grading: 1 point for expansion, 1 point for answer.

  (b) (2 points) $\sum_{i=1}^{2}\sum_{j=1}^{3}(i+j)$

  $\sum_{i=1}^{2}\sum_{j=1}^{3}(i+j)$
  $= \sum_{i=1}^{2}((i+1)+(i+2)+(i+3))$
  $= \sum_{i=1}^{2}(3i+6)$
  $= (3\cdot 1+6)+(3\cdot 2+6)$
  $= 21$

  Grading: 1 point for steps, 1 point for answer (i.e., -1 if no steps).

(c) (4 points) $\sum_{i=1}^{50}\sum_{j=i}^{50} ij$

For this problem, we first find a general formula. Let $n = 50$.

$$\sum_{i=1}^{n}\sum_{j=i}^{n} ij$$

$$= \sum_{i=1}^{n}\left(i \cdot \sum_{j=i}^{n} j\right)$$

$$= \sum_{i=1}^{n}\left(i \cdot \left(\sum_{j=1}^{n} j - \sum_{j=1}^{i-1} j\right)\right)$$

$$= \sum_{i=1}^{n}\left(i \cdot \left(\frac{n(n+1)}{2} - \frac{(i-1)i}{2}\right)\right)$$

$$= \frac{1}{2}\sum_{i=1}^{n}\left(n(n+1)i - (i-1)i^2\right)$$

$$= \frac{1}{2}\sum_{i=1}^{n}\left(n(n+1)i - i^3 + i^2\right)$$

$$= \frac{1}{2}\left(n(n+1)\sum_{i=1}^{n} i - \sum_{i=1}^{n} i^3 + \sum_{i=1}^{n} i^2\right)$$

$$= \frac{1}{2}\left(\frac{n^2(n+1)^2}{2} - \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}\right)$$

$$= \frac{1}{2}\left(\frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}\right)$$

$$= \frac{n(n+1)(3n(n+1) + 2(2n+1))}{24}$$

Now, substitute $n = 50$,

$$\sum_{i=1}^{50}\sum_{j=i}^{50} ij$$

$$= \frac{50 \cdot 51 \cdot (3 \cdot 50 \cdot (51) + 2 \cdot (101))}{24}$$

$$= 834275$$

Grading: 3 points for steps, 1 point for answer. It is not necessary to substitute n for 50, they may have 50 in place when solving the sum. Calculating the solution with a program is not allowed (-3).

- **Problem 2** (5 points) Let $f(x) = \dfrac{3x}{x+6}$, for all real $x \geq 1$. Find $f^{-1}(x)$ and state the domain and range of $f^{-1}(x)$.

  Let $y = f^{-1}(x)$. By definition of inverse,

$$
\begin{aligned}
f(f^{-1}(x)) &= x \\
f(y) &= x \\
\frac{3y}{y+6} &= x \\
3y &= x(y+6) \\
3y &= xy + 6x \\
y(3-x) &= 6x \\
y &= \frac{6x}{3-x} \\
f^{-1}(x) &= \frac{6x}{3-x}
\end{aligned}
$$

  Next, we want to know the range of f. When $x = 1$, $f(x) = \frac{3}{7}$, and as x increases, $f(x)$ increases as well, but $\lim_{x\to\infty} f(x) = 3$. So, the range of f is $\{x : x \in \mathbb{R} \text{ and } \frac{3}{7} \leq x < 3\}$.

  Then, domain(f^{-1}) = range(f) = $\{x : x \in \mathbb{R} \text{ and } \frac{3}{7} \leq x < 3\}$. And, range$(f^{-1})$ = domain(f) = $\{x : x \in \mathbb{R} \text{ and } x \geq 1\}$.

  Grading: 3 points for inverse, 1 point for domain, 1 point for range.

- **Problem 3** Let A, B, C be finite sets and f, g be functions such that $f : A \to B$ and $g : B \to C$.

  (a) (4 points) Give an example where f is surjective, g is injective, but $g \circ f$ is neither.

  Consider the following example. Let $A = \{1, 2\}, B = \{3\}, C = \{4, 5\}$ and define $f : A \to B$ as $f(1) = f(2) = 3$ and $g : B \to C$ as $g(3) = 4$. f is surjective and g is injective. Then, $(g \circ f) : A \to C$ is the function where $(g \circ f)(1) = (g \circ f)(2) = 4$, which is neither surjective nor injective.

  Grading: 2 points for a valid example, 2 points for a valid argument. (give 1 total if example is incorrect).

  (b) (4 points) Prove that if $g \circ f$ is injective, then f is injective.

  Let $g \circ f$ be injective. Suppose $f(x) = f(y)$,

$$
\begin{aligned}
f(x) &= f(y) \\
g(f(x)) &= g(f(y)) && (g \text{ is a function}) \\
(g \circ f)(x) &= (g \circ f)(y) && (\text{definition of composition of function}) \\
x &= y && (g \circ f \text{ is a injective})
\end{aligned}
$$

  So, $f(x) = f(y) \to x = y$, this means f is injective.

  Grading: 4 points if proof is correct, 1 point if proof is incorrect but attempted problem.

- **Problem 4** (3 points) Apply Euclidean Algorithm to find $\gcd(546, 495)$.

$$
\begin{aligned}
&\gcd(546, 495) & \\
&= \gcd(495, 51) & (546 = (1) \cdot 495 + (51)) \\
&= \gcd(51, 36) & (495 = (9) \cdot 51 + (36)) \\
&= \gcd(36, 15) & (51 = (1) \cdot 36 + (15)) \\
&= \gcd(15, 6) & (36 = (2) \cdot 15 + (6)) \\
&= \gcd(6, 3) & (15 = (2) \cdot 6 + (3)) \\
&= \gcd(3, 0) & (6 = (2) \cdot 3 + (0)) \\
&= 3 &
\end{aligned}
$$

Grading: 2 points for steps, 1 point for answer. (give 1 total if step is incorrect which leads to an incorrect answer, but attempted the problem). Only give 1 point if solution is correct, but does not use Euclidean Algorithm.

- **Problem 5** (6 points) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions where $f(x) = x^2 + 1$ and $g(x) = x + 2$. Find $f \circ g$ and $g \circ f$.

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f(x + 2) \\
&= (x + 2)^2 + 1
\end{aligned}
$$

$$
\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g(x^2 + 1) \\
&= (x^2 + 1) + 2 \\
&= x^2 + 3
\end{aligned}
$$

Grading: 3 points each. 2 points for steps, 1 point for correct final expression.