COT 3100 Discrete Mathematics
Homework 2 Key

February 19, 2010

Problem 1.

Use Laws of Logic and Rules of Inference to justify the following arguments.

(a) Section 1.5 #28, 5 pts

$$\begin{array}{l} \forall x(P(x) \vee Q(x)) \\ \forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x)) \\ \hline \therefore \forall x(\neg R(x) \rightarrow P(x)) \end{array}$$

	Step	**Reason**
1.	$\forall x(P(x) \vee Q(x))$	Premise
2.	$P(a) \vee Q(a)$	Universal instantiation from (1)
3.	$\forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x))$	Premise
4.	$\forall x(\neg(\neg P(x) \wedge Q(x))) \vee R(x))$	Rule of Implication from (3)
5.	$\forall x((P(x) \vee \neg Q(x)) \vee R(x))$	DeMorgan's Law from (4)
6.	$P(a) \vee \neg Q(a) \vee R(a)$	Universal instantiation from (5)
7.	$P(a) \vee P(a) \vee R(a)$	Resolution from (2) and (6)
8.	$P(a) \vee R(a)$	Idempotent Law from (7)
9.	$\neg R(a) \rightarrow P(a)$	Rule of Implication from (8)
10.	$\forall x(\neg R(x) \rightarrow P(x))$	Universal generalization from (9)

(b) 5 pts

$$\begin{array}{l} p \vee q \\ u \wedge r \\ r \rightarrow \neg t \\ (s \vee p) \rightarrow t \\ \hline \therefore q \end{array}$$

	Step	**Reason**
1.	$u \wedge r$	Premise
2.	r	Simplification using (1)
3.	$r \rightarrow \neg t$	Premise
4.	$\neg t$	Modus ponens using (2) and (3)
5.	$(s \vee p) \rightarrow t$	Premise
6.	$\neg(s \vee p)$	Modus tollens using (4) and (5)
7.	$\neg s \wedge \neg p$	DeMorgan's Law using (6)
8.	$\neg p$	Simplification using (7)
9.	$p \vee q$	Premise
10.	q	Disjunctive syllogism using (8) and (9)

Grading:

1. Full credit if everything follows correctly *with reasons.*
2. -2 points each if reasons are missing, or missing more than half of the steps.
3. -3 points each if the solution used verbal arguments instead of algebra (logic and inference).
4. Give 0 point each if there exists no procedure but conclusion.

Problem 2.

Prove or disprove each of the following statements.

(a) (Section 1.6 #8, 5 pts) If $n \geq 1$ is a perfect square, then $n+2$ is not a perfect square.

Proof: Since $n \geq 1$ is a perfect square
There exists an integer $m \geq 1$ so that $n = m^2$
Therefore the smallest perfect square greater than n is $(m+1)^2$
$(m+1)^2 - (n+2) = m^2 + 2m + 1 - (n+2) = n + 2m + 1 - n - 2 = 2(m-1) + 1$
Since $m \geq 1$, thus $2(m-1) + 1 > 0$
Therefore $(m+1)^2 > (n+2)$, and n + 2 cannot be a perfect square.

(b) (Section 1.7 #10, 5 pts) Consider the following numbers.

$65^{1006} - 8^{2001} + 3^{177}$
$79^{1210} - 9^{2399} + 2^{2001}$
$24^{4491} - 5^{8190} + 7^{1775}$

It is possible to select 2 different numbers from the 3 numbers above such that their product is non-negative.

Proof: Of these three numbers, at least two must have the same sign (both non-negative or both negative), since there are only two signs (negative and non-negative) need to be considered.
The product of two with the same sign is non-negative.
It is a non-constructive proof, since we have not identified which product is non-negative.

(c) (Section 1.7 #12, 5 pts) If a and b are rational numbers, then a^b is also rational.

Disproof: Take $a = 2$ and $b = 1/2$,
Then $a^b = 2^{1/2} = \sqrt{2}$
Because we know that $\sqrt{2}$ is not rational, then we disprove the statement.

(d) (Section 1.7 #32, 5 pts) $\sqrt[3]{2}$ is irrational.

Proof: We can prove it by contradiction, which means that $\sqrt[3]{2}$ is rational.
Then we have $\sqrt[3]{2} = a/b$, where a and b are integers without common factors.
It follows that $2 = a^3/b^3$, hence, $2b^3 = a^3$.
By the definition of an even integer it follows that a^3 is even,
therefore a must also be even.
Furthermore we can let $a = 2c$ for some integer c, thus $2b^3 = 8c^3$.
After dividing both sides of this equation by 2 gives $b^3 = 4c^3$,
which means b^3 is even, again b must be even as well.
We have now concluded that a and b are both even, thus 2 is a common divisor of a and b.
This contradicts the choice of a/b.
Therefore the assumption—that $\sqrt[3]{2}$ is rational—is in error,
so we have proved that $\sqrt[3]{2}$ is irrational.

Grading:

1. -2 each if answer is correct but justification is incorrect.
2. -3 each if answer is correct but missing justification.
3. Give 0 point if answer is incorrect.

Problem 3.

Let A, B and C be sets and let $P(X)$ be the powerset of set X. Prove or disprove the following statements.

(a) (5 pts) If $A \subseteq (B \cup C)$, then $A \subseteq B$ or $A \subseteq C$.

Disproof: Take $A = \{1,2\}, B = \{1,3\}$ and $C = \{2,4\}$.
Then $A \subseteq B \cup C = \{1,2,3,4\}$, but A is neither a subset of B, nor a subset of C

(b) (5 pts) $(A - C) \cap (C - B) = \emptyset$.

Proof: We can prove it by contradiction.
Assume that$(A - C) \cap (C - B) \neq \emptyset$ to show that it results to contradiction.
$(A - C) \cap (C - B) \neq \emptyset$ means that there exists some $x \in (A - C) \cap (C - B)$.
By the definition of intersection we can imply, that there exists x for which the following proposition is true: $p = (x \in A - C) \wedge (x \in C - B)$.
Using the definition of set difference we can rewrite p as:
$p = (x \in A) \wedge (x \notin C) \wedge (x \in C) \wedge (x \notin B)$.
But $(x \notin C) \wedge (x \in C) =$ False, so
$p = (x \in A) \wedge [(x \notin C) \wedge (x \in C)] \wedge (x \notin B) = (x \in A) \wedge$ False $\wedge (x \notin B) =$ False.
Thus, the assumption that intersection $(A - C) \cap (C - B) \neq \emptyset$ is not empty results to contradiction which proves that this assumption is false, i.e. intersection is empty.

(c) (5 pts) $P(A) - P(B) \subseteq P(A - B)$.

Disproof: Take a counterexample: $A = \{1,2\}, B = \{2,3\}$,
$P(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}, P(B) = \{\emptyset, \{2\}, \{3\}, \{2,3\}\}$
$P(A) - P(B) = \{\{1\}, \{1,2\}\}, A - B = \{1\}, P(A - B) = \{\emptyset, \{1\}\}$
Thus, $\{1,2\} \in P(A) - P(B)$, but $\{1,2\} \notin P(A - B)$
so the proposition $P(A) - P(B) \subseteq P(A - B)$ is disproved.

Grading:

1. -2 each if answer is correct but justification is incorrect.
2. -3 each if answer is correct but missing justification.
3. Give 0 point if answer is incorrect.