COT 3100 Discrete Mathematics
Homework 1 Key

February 5, 2010

Problem 1

Section 1.2 #10. Show that each of these conditional statements is a tautology by using truth tables.

(a) (1 point) $[\neg p \wedge (p \vee q)] \rightarrow q$

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

The expression $[\neg p \wedge (p \vee q)] \rightarrow q$ is a tautology because it evaluates to true for all values of p and q.

(b) (1 point) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

The expression $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ is a tautology because it evaluates to true for all values of p, q and r. Note that this is hypothetical syllogism.

(c) (1 point) $[p \wedge (p \rightarrow q)] \rightarrow q$

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

The expression $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology because it evaluates to true for all values of p and q. Note that this is modus ponens.

(d) (1 point) $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \vee q) \wedge (p \to r) \wedge (q \to r)$	$[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

The expression $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$ is a tautology because it evaluates to true for all values of p, q and r.

Grading:

1. 1 point per table.
2. Give 1 point if everything is correct and has all columns.
3. Give 1/2 point if minor mistakes, missing two or more columns, or missing one or more rows.
4. If they did the wrong problems (in particular, if they copied the ones presented in the Lab), then give 0 points.

Section 1.2 #12. Show that each conditional statement is tautology without using truth tables. Use logical equivalences (on P.24 and P.25 of textbook)

(a) (1 point)

$[\neg p \wedge (p \vee q)] \to q$
$\equiv [(\neg p \wedge p) \vee (\neg p \wedge q)] \to q$ (Distributive Law)
$\equiv [\mathrm{F} \vee (\neg p \wedge q)] \to q$ (Negation Law)
$\equiv (\neg p \wedge q) \to q$ (Identity Law)
$\equiv \neg(\neg p \wedge q) \vee q$ (Rule of Implication)
$\equiv (\neg(\neg p) \vee \neg q) \vee q$ (DeMorgan's Law)
$\equiv (p \vee \neg q) \vee q$ (Law of Double Negation)
$\equiv p \vee (\neg q \vee q)$ (Associative Law)
$\equiv p \vee \mathrm{T}$ (Negation Law)
$\equiv \mathrm{T}$ (Domination Law)

(b) (2 points)

$[(p \to q) \land (q \to r)] \to (p \to r)$	
$\equiv [(\neg p \lor q) \land (\neg q \lor r)] \to (\neg p \lor r)$	(Rule of Implication)
$\equiv \neg[(\neg p \lor q) \land (\neg q \lor r)] \lor (\neg p \lor r)$	(Rule of Implication)
$\equiv [\neg(\neg p \lor q) \lor \neg(\neg q \lor r)] \lor (\neg p \lor r)$	(DeMorgan's Law)
$\equiv [(\neg(\neg p) \land \neg q) \lor (\neg(\neg q) \land \neg r)] \lor (\neg p \lor r)$	(DeMorgan's Law)
$\equiv [(p \land \neg q) \lor (q \land \neg r)] \lor (\neg p \lor r)$	(Law of Double Negation)
$\equiv (p \land \neg q) \lor (q \land \neg r) \lor (\neg p) \lor r$	(Associative Law)
$\equiv (\neg p) \lor (p \land \neg q) \lor r \lor (q \land \neg r)$	(Commutative Law)
$\equiv (\neg p \lor (p \land \neg q)) \lor (r \lor (q \land \neg r))$	(Associative Law)
$\equiv [(\neg p \lor p) \land (\neg p \lor \neg q)] \lor [(r \lor q) \land (r \lor \neg r)]$	(Distributive Law)
$\equiv [\mathrm{T} \land (\neg p \lor \neg q)] \lor [(r \lor q) \land \mathrm{T}]$	(Negative Law)
$\equiv (\neg p \lor \neg q) \lor (r \lor q)$	(Identity Law)
$\equiv (\neg p) \lor (\neg q) \lor r \lor q$	(Assocaitive Law)
$\equiv (\neg p) \lor r \lor (\neg q) \lor q$	(Commutative Law)
$\equiv (\neg p \lor r) \lor (\neg q \lor q)$	(Assocaitive Law)
$\equiv (\neg p \lor r) \lor \mathrm{T}$	(Negation Law)
$\equiv \mathrm{T}$	(Domination Law)

(c) (1 point)

$[p \land (p \to q)] \to q$	
$\equiv [p \land (\neg p \lor q)] \to q$	(Rule of Implication)
$\equiv [(p \land \neg p) \lor (p \land q)] \to q$	(Distributive Law)
$\equiv [\mathrm{F} \lor (p \land q)] \to q$	(Negation Law)
$\equiv (p \land q) \to q$	(Identity)
$\equiv \neg(p \land q) \lor q$	(Rule of Implication)
$\equiv (\neg p \lor \neg q) \lor q$	(DeMorgan's Law)
$\equiv (\neg p) \lor (\neg q \lor q)$	(Assocaitive Law)
$\equiv (\neg p) \lor \mathrm{T}$	(Negation Law)
$\equiv \mathrm{T}$	(Domination Law)

(d) (2 points)

$[(p \lor q) \land (p \to r) \land (q \to r)] \to r$	
$\equiv [(p \lor q) \land (\neg p \lor r) \land (\neg q \lor r)] \to r$	(Rule of Implication)
$\equiv \neg[(p \lor q) \land (\neg p \lor r) \land (\neg q \lor r)] \lor r$	(Rule of Implication)
$\equiv [\neg(p \lor q) \lor \neg(\neg p \lor r) \lor \neg(\neg q \lor r)] \lor r$	(DeMorgan's Law)
$\equiv [(\neg p \land \neg q) \lor (\neg(\neg p) \land \neg r) \lor (\neg(\neg q) \land \neg r)] \lor r$	(DeMorgan's Law)
$\equiv [(\neg p \land \neg q) \lor (p \land \neg r) \lor (q \land \neg r)] \lor r$	(Law of Double Negation)
$\equiv (\neg p \land \neg q) \lor (p \land \neg r) \lor [(q \land \neg r) \lor r]$	(Assocaitive Law)
$\equiv (\neg p \land \neg q) \lor (p \land \neg r) \lor [(q \lor r) \land (\neg r \lor r)]$	(Distributive Law)
$\equiv (\neg p \land \neg q) \lor (p \land \neg r) \lor [(q \lor r) \land \mathrm{T}]$	(Negation Law)
$\equiv (\neg p \land \neg q) \lor (p \land \neg r) \lor (q \lor r)$	(Identity Law)
$\equiv (\neg p \land \neg q) \lor (p \land \neg r) \lor q \lor r$	(Associative Law)
$\equiv (\neg p \land \neg q) \lor q \lor (p \land \neg r) \lor r$	(Commutative Law)
$\equiv [(\neg p \land \neg q) \lor q] \lor [(p \land \neg r) \lor r]$	(Associative Law)
$\equiv [(\neg p \lor q) \land (\neg q \lor q)] \lor [(p \lor r) \land (\neg r \lor r)]$	(Distributive Law)
$\equiv [(\neg p \lor q) \land \mathrm{T}] \lor [(p \lor r) \land \mathrm{T}]$	(Negation Law)
$\equiv (\neg p \lor q) \lor (p \lor r)$	(Identity)
$\equiv \neg p \lor q \lor p \lor r$	(Associative Law)
$\equiv \neg p \lor p \lor q \lor r$	(Commutative Law)
$\equiv (\neg p \lor p) \lor (q \lor r)$	(Associative Law)
$\equiv \mathrm{T} \lor (q \lor r)$	(Negation Law)
$\equiv \mathrm{T}$	(Domination Law)

Grading:

1. Full credit if everything follows correctly *with reasons.*
2. Half credit (1/2 point for (a, c), 1 point for (b, d)) if reasons are missing, or missing more than half of the steps.
3. -2 point overall if the solution used verbal arguments instead of algebra (logical equivalences).
4. Give 0 points for (b) and (c) if solution directed applied hypothetical syllogism or modus ponens, because that is exactly what is asked to be proven.

Problem 2

Section 1.3 #32. Express each of these statements using quantifiers. Then form the negation of the statement so that no negation sign is to the left of a quantifier. Next, express the negation in simple English.

(a) (2 points) All dogs have fleas.

Original sentence	All dogs have fleas.
Predicate and domain	Let $P(x)$ be the predicate "x has fleas", where the domain is the set of dogs.
Logical expression	$\forall x P(x)$
Negated logical expression	$\exists x \neg P(x)$
Negated sentence	There is a dog that does not have fleas.

(b) (2 points) There is a horse that can add.

Original sentence	There is a horse that can add.
Predicate and domain	Let $P(x)$ be the predicate "x can add", where the domain is the set of horses.
Logical expression	$\exists x P(x)$
Negated logical expression	$\forall x \neg P(x)$
Negated sentence	No horse can add.

(c) (2 points) Every koala can climb.

Original sentence	Every koala can climb.
Predicate and domain	Let $P(x)$ be the predicate "x can climb", where the domain is the set of koalas.
Logical expression	$\forall x P(x)$
Negated logical expression	$\exists x \neg P(x)$
Negated sentence	There is a koala that cannot climb.

(d) (2 points) No monkey can speak French.

Original sentence	No monkey can speak French.
Predicate and domain	Let $P(x)$ be the predicate "x can speak French", where the domain is the set of monkeys.
Logical expression	$\forall x \neg P(x)$ (or $\neg \exists x P(x)$)
Negated logical expression	$\exists x P(x)$
Negated sentence	There is a monkey that can speak French.

(e) (2 points) There exists a pig that can swim and catch fish.

Original sentence	There exists a pig that can swim and catch fish.
Predicate and domain	Let $P(x)$ be the predicate that "x can swim" and $Q(x)$ be the predicate that "x can catch fish", where the domains of both predicates are over the set of pigs.
Logical expression	$\exists x(P(x) \wedge Q(x))$
Negated logical expression	$\neg(\exists x(P(x) \wedge Q(x)))$ $\equiv \forall x \neg(P(x) \wedge Q(x))$ $\equiv \forall x(\neg P(x) \vee \neg Q(x))$
Negated sentence	Every pig either cannot swim or cannot catch fish, or both.

Grading:

1. 2 points each.
2. -1 if the solution did not translate the english sentence into logical expression.
3. -1 if the final sentence is correct, but the negated logical expression is incorrect.
4. $-1/2$ if solution did not specify domains.
5. $-1/2$ for other minor mistakes.
6. Note that there are multiple correct answers. For example, in (e), you can have the negated expression being $\forall x \neg(P(x) \wedge Q(x))$, and the corresponding sentence will be "No pigs can both swim and catch fish".
7. Using different variable names are, of course, allowed.

Problem 3

Section 1.4 #28. Determine the truth value of these statements if the domain of each variable consists of all real numbers.

Note that the notation $\mathbb{R}$ denotes the set of real numbers.

(a) $\forall x \exists y(x^2 = y)$.

True. Given any value for x, compute x^2 and let this value be y. Then, $y = x^2$. This is valid because if $x \in \mathbb{R}$, then $x^2 \in \mathbb{R}$.

(b) $\forall x \exists y(x = y^2)$.

False. If $x = -1$, then there is no value of y such that $y^2 = x$ holds because $y^2 \geq 0 > x$ for all $y \in \mathbb{R}$.

(c) $\exists x \forall y(xy = 0)$.

True. Let $x = 0$, then $xy = 0 \times y = 0$ for any $y \in \mathbb{R}$.

(d) $\exists x \exists y(x + y \neq y + x)$.

False. $x + y = y + x$ for all $x, y \in \mathbb{R}$. This is the commutative law for addition of real numbers.

(e) $\forall x(x \neq 0 \rightarrow \exists y(xy = 1))$

True. Given any value of x, if $x \neq 0$, then we may let $y = \frac{1}{x}$, which is a real number, and $xy = x \times \frac{1}{x} = 1$.

(f) $\exists x \forall y(y \neq 0 \rightarrow xy = 1)$

False. This one is more interesting. To show that the statement is false, we can show that its negation, $\forall x \exists y(y \neq 0 \wedge xy \neq 1)$ is true.

Let x be given. There are two cases.

1. $x = 0$. Let $y = 1$, then $y \neq 0$ and $xy = 0 \cdot 1 = 0 \neq 1$.
2. $x \neq 0$. Let $y = \frac{2}{x}$, then $y \neq 0$ and $xy = x \cdot \frac{2}{x} = 2 \neq 1$.

(g) $\forall x \exists y(x + y = 1)$

True. Given any value of x, let $y = 1 - x$. Then, $x + y = x + (1 - x) = 1$.

(h) $\exists x \exists y(x + 2y = 2 \land 2x + 4y = 5)$

False. For any value of x and y, if $x + 2y = 2$ is true, then $2 \times (x + 2y) = 2 \times 2$. Which means $2x + 4y = 4$ is true, so $2x + 4y \neq 5$. The two equations are inconsistent.

(i) $\forall x \exists y(x + y = 2 \land 2x - y = 1)$

False. Similar to (f), the easy way around is to show the negation, $S = \exists x \forall y(x + y \neq 2 \lor 2x - y \neq 1)$ is true. Applying rule of implication, $(x + y \neq 2 \lor 2x - y \neq 1) \equiv (x + y = 2 \rightarrow 2x - y \neq 1)$. Thus, $S = \exists x \forall y(x + y = 2 \rightarrow 2x - y \neq 1)$.

Let $x = 0$. Then, let y be given. If $x + y = 2$, then $y = 2$, and, $2x - y = 2 \cdot 0 - 2 = -2 \neq 1$. Thus, S is true, and the original statement is false.

(j) $\forall x \forall y \exists z(z = (x + y)/2)$

True. Let x and y be given. Then, let $z = (x + y)/2$. Note that $z \in \mathbb{R}$.

Grading:

1. 1 point each.
2. -1 if answer is incorrect. Note that if the answer is incorrect, any justification of it is bound to be incorrect as well.
3. $-1/2$ if answer is correct, but justification is missing or incorrect.
4. Shortened form of justification is allowed. (For example, in (b) you may simply say "if x is negative, then no y can satisfy the predicate")
5. There are multiple ways to justify an answer. (For example, in (i), you may say the only solution to the system of equations is when $x = y = 1$, thus the statement cannot be true for all $x \in \mathbb{R}$).

Problem 4

Section 1.4 #34. Find a common domain for the variables x, y, and z for which the statement $\forall x \forall y((x \neq y) \rightarrow \forall z((z = x) \lor (z = y)))$ is true and another domain for which it is false.

(2 points) For the statement to be *true*, let the domain be the set $\{0, 1\}$. Then, for any given values of x and y, if $x \neq y$, then one of them is 0 and the other is 1. Then for all z, $z \in \{0, 1\}$, $(z = x) \lor (z = y)$. In fact, the statement is *true* for any domain with cardinality at most two (even if domain is empty).

Lab Section #1 students: If we have a set $S = \{0\}$, and x, y, z all belong to S, the statement will be true as well. Since the hypothesis is always false for all values of x and y, the conditional statement will be always true.

(2 points) If the domain is the set of integers, then the statement is *false*. To show this, consider the negation, $\exists x \exists y(x \neq y \land \exists z(z \neq x \land z \neq y))$. Now, let $x = 1, y = 2$. $x \neq y$ and let $z = 3$, then $z \neq x$ and $z \neq y$. In fact, the original statement is *false* whenever the domain contains at least 3 elements.

Grading:

1. There are 2 parts here, 2 points each.
2. -1 point per for missing justification in each part.
3. $-1/2$ per if the domain is correct but justification is incorrect.
4. -2 per for incorrect domain.

Section 1.4 #46. Determine the truth value of the statement $\exists x \forall y (x \leq y^2)$ if the domain for the variables consists of

(a) (2 points) the positive real numbers. *False.* Consider the negation, $\forall x \exists y (x > y^2)$. We want to show that the negation is true. Let x be given, then let $y = \sqrt{\frac{x}{2}}$. y is a positive real number and $y^2 = (\sqrt{\frac{x}{2}})^2 = \frac{x}{2} < x$. Note that the last inequality holds because x is positive.

(b) (2 points) the integers. *True.* Let $x = -1$, then for all y, $x \leq 0 \leq y^2$.

(c) (2 points) the nonzero real numbers. *True.* Let $x = -1$, then for all y, $x \leq 0 \leq y^2$.

Grading:

1. 2 points each.
2. -1 each if missing justification.
3. $-1/2$ each if answer is correct, but justification is incorrect.