(20 marks) Problem 1.

Prove that the sum of the squares of any four consecutive positive integers always leaves a remainder of six when divided by eight.

Proof by Induction: (3 marks) The statement says

$$\sum_{i=0}^{3}(n+i)^2 \equiv 6\ (mod\ 8)$$

(3 marks) Base case: n = 1;

$$\sum_{i=0}^{3}(1+i)^2 \equiv 1^2+2^2+3^2+\ 4^2 \equiv 1+4+9+16\ (mod\ 8) \equiv 1+4+1+0\ (mod\ 8) \equiv 6\ (mod\ 8)$$

(3 marks) Inductive Hypothesis:

$$\sum_{i=0}^{3}(k+i)^2 \equiv 6\ (mod\ 8)\ for\ 1 \leq k < n$$

Induction:

(3 marks)

$$\textstyle\sum_{i=0}^{3}(n+i)^2 \equiv n^2+(n+1)^2+(n+2)^2+\ (n+3)^2(mod\ 8)$$

(4 marks) $\equiv (n-1)^2+n^2+(n+1)^2+(n+2)^2+\ (n+3)^2-(n-1)^2(mod\ 8)$

$$\left.\begin{aligned}
&\equiv \textstyle\sum_{i=0}^{3}(n-1+i)^2 \quad +(n+3)^2-(n-1)^2(mod\ 8)\\
&\equiv 6+n^2+6n+9-n^2+\ 2n-1(mod\ 8)\\
&\equiv 6+8n+8(mod\ 8)\\
&\equiv 6(mod\ 8)
\end{aligned}\right\}\ \text{(4 marks)}$$

(20 marks) Problem 2. The Lucas numbers, L_i are defined as

$$\begin{cases} L_1 = 1 \\ L_2 = 3 \\ L_i = L_{i-1} + L_{i-2} \text{ for } i \geq 3 \end{cases}$$

Thus, $L_3 = 4$ and $L_4 = 7$. Prove by mathematical induction, that for $n \geq 1$,

$$\sum_{i=1}^{n} L_i^{\,2} = L_n L_{n+1} - 2$$

(4 points) Base case, n = 1: $\sum_{i=1}^{n} L_i^{\,2} = L_1^{\,2} = 1$ (L.H.S)

$$L_n L_{n+1} - 2 = L_1 \cdot L_2 - 2 = 1 \cdot 3 - 2 = 1 \text{ (R.H.S)}$$

(4 points) Inductive Hypothesis:

$\sum_{i=1}^{k} L_i^{\,2} = L_k L_{k+1} - 2$ for $1 \leq k < n$

Induction:

$$\sum_{i=1}^{n} L_i^{\,2} = \sum_{i=1}^{n-1} L_i^{\,2} + L_n^{\,2}$$ (2 points for breaking up the sum)

$= L_{n-1} L_n - 2 + L_n^{\,2}$ (5 points for applying I.H.)

$= L_n (L_{n-1} + L_n) - 2$

$= L_n L_{n+1} - 2$ (5 points for complete solution)

Problem 3

Apply Euclidian Algorithm to find gcd(481,592)

592 = 1·481 + 111 (5 points)
481 = 4·111+37 (5 points)
111 = 3·37 + 0 (5 points)
gcd(481, 592) = 37 (5 points for identifying the GCD correctly)

Problem 4

Use Laws of Logic and Rules of Inference to justify the following argument

$r \vee q$
$r \rightarrow t$
$q \rightarrow s$
$(t \vee s) \rightarrow v$
$v \rightarrow (u \wedge p)$

$\therefore p$

1.	$r \vee q$	Premise	
2.	$\neg q \rightarrow r$	Law of implication of 1	(2 points)
3.	$r \rightarrow t$	Premise	
4.	$\neg q \rightarrow t$	Hypothetical Syllogism of 2,3	(3 points)
5.	$q \rightarrow s$	Premise	
6.	$q \vee t$	Law of implication of 4	(2 points)
7.	$\neg q \vee s$	Law of implication of 5	(2 points)
8.	$t \vee s$	Resolution	(3 points)
9.	$(t \vee s) \rightarrow v$	Premise	
10.	v	Modus Ponens from 8,9	(3 points)
11.	$v \rightarrow (u \wedge p)$	Premise	
12.	$u \wedge p$	Modus Ponens from 10,11	(3 points)
13.	p	Simplification on 12	(2 points)

Problem 5

Let $f: A \rightarrow B$ and $g: B \rightarrow A$ be functions such that

(i) $\forall y \in B \ (f \circ g)(y) = y$
(ii) $\exists x \in A \ (g \circ f)(x) \neq x$

Prove that f is surjective, but not injective.

f is surjective: $\forall y \in B \ \exists x \in A : f(x) = y$ (4 points for definition of surjective, verbal OK)
Let $y \in B$, then let $x = g(y) \in A$ (2 points for proper start of proof)
$f(x)$
$= f(g(y))$ (By the choice/definition of x)
$= (f \circ g)(y)$ (Definition of function composition)
$= y$ (By Condition (i))
$\therefore \forall y \in B, \exists x \in A$, in particular $g(y) \in A$: $f(x) = y$. (4 points for complete solution)

f is not injective:

$\neg(\forall a_1, a_2 \in A, f(a_1) = f(a_2) \rightarrow a_1 = a_2)$ (4 points for definition of injective, verbal OK)

$\exists\ a_1, a_2 \in A: f(a_1) = f(a_2) \wedge a_1 \neq a_2$ (2 points for negation)

Apply existential instantiation on (ii)

Let a_1 be such that $(g \circ f)(a_1) \neq a_1$

$a_1 \in A$, and $(g \circ f)(a_1) \in A$

Let $a_2 = (g \circ f)(a_1)$. $a_2 \in A$ and $a_1 \neq a_2$

$f(a_2)$

$= f((g \circ f)(a_1))$ (By the choice of a_2)

$= f(g(f(a_1)))$ (Definition of $g \circ f$)

$= (f \circ g)(f(a_1))$ (Definition of $f \circ g$)

$= f(a_1)$ (By Condition (i) with $y = f(a_1) \in B$).

$\therefore \exists\ a_1, a_2 : a_1 \neq a_2 \wedge f(a_1) = f(a_2)$

(4 points for complete solution)

Extra Credit

Find c such that $1000! = 9^c \times k$ and $9 \nmid k$, for integers c and k.

Let $f_3(n)$ be the number of factors of 3 in n!

$$\text{Then } f_3(n) = \begin{cases} 0 & \text{if } n < 3 \\ \lfloor n/3 \rfloor + f_3(\lfloor n/3 \rfloor) & \text{otherwise} \end{cases}$$

(6 points for correct recurrence)

Just calculations from this point.

$$\begin{aligned} f_3(1000) &= f_3(333) + 333 \\ &= f_3(111) + 111 + 333 \\ &= f_3(37) + 37 + 444 \\ &= f_3(12) + 12 + 481 \\ &= f_3(4) + 4 + 493 \\ &= f_3(1) + 1 + 497 \\ &= 0 + 498 \end{aligned}$$

$\therefore 1000! = 3^{498} \cdot k = 9^{249} \cdot k$

$c = 249$

(4 points for correct answer)