COT 3100 Final Exam - Part A (Relations, Functions) - 25 pts (12/3/2024) Solutions

1) (5 pts) How many anti-symmetric relations over the set A = {1, 2, 3, 4, 5, 6, 7, 8} contain the ordered pairs (1, 1), (2, 3), (6, 4), (7, 7), and (8, 3)? Please leave your answer as a product of terms, each of which is written in exponential form.

We must include (1, 1) and (7, 7). We can either include or not include (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) and (8, 8). We can make these selections in 2^6 ways for these 6 ordered pairs.

We are not allowed to include (3, 2), (4, 6) and (3, 8), because if we did, the relation would not be anti-symmetric.

Consider the other 25 ordered pairs of pairs of the form ((a, b), (b, a)) where a ≠ b. (To figure out that there are 25 of these, note that we start with 8 x 8 = 64 possible ordered pairs in the relation. Then we subtract out the 8 of the form (a, a) to get 64 – 8 = 56. Next, we subtract out the pairs (2, 3), (3, 2), (6, 4), (4, 6), (8, 3) and (3, 8) because we've already accounted for the membership of these ordered pairs. This leaves us with 56 – 6 = 50 ordered pairs, which we put in 25 groups, each with 2 ordered pairs. For each of these 25 groups, we can choose one of three things: put nothing into our relation, put (a, b) in our relation or put (b, a) in our relation. (We're not allowed to put both (a, b) and (b, a) in the relation and maintain anti-symmetry. Thus, we can make our choices of inclusion on these 50 ordered pairs in 3^{25} ways.

Since any subset from the first part can be paired with any subset from the second part, we multiply the number of choices from both parts to get a final answer of $\mathbf{2^6 3^{25}}$.

$\mathbf{2^6 3^{25}}$ **(Grading: 2 pts first term, 2 pts second term, 1 pt mult both terms)**

2) (6 pts) Let R = { (a, b) | $\exists c \in Z^+$ $such\ that\ ab = c^2$} be a relation defined over the positive integers. Prove that R is transitive. (You may use the fact that if a rational number squared is an integer, then that rational number itself is also an integer.)

We must show that if $(a, b) \in R \wedge (b, c) \in R$, then $(a, c) \in R$. **(Grading: 1 pt)**

We use direct proof. Let $(a, b) \in R \wedge (b, c) \in R$. This means that there are integers d and e such that: $ab = d^2$ and $bc = e^2$. **(Grading: 1 pt)** Multiply these equations together to get:

$(ab)(bc) = d^2e^2$
$(ac)b^2 = d^2e^2$
$ac = \frac{(de)^2}{b^2} = (\frac{de}{b})^2$, **(Grading: 2 pts)**

At this point, note that the RHS is a rational number squared. But because a and c are positive integers, it's also an integer. Using the fact given, this means that the rational number that is being squared is in fact an integer. We can conclude that $\frac{de}{b} \in Z$, thus by definition of R, $(a, c) \in R$. **(Grading: 2 pts for proper communication)**

3) (8 pts) Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be functions where f is a surjective function and g is an injective function.

(a) Prove or disprove: the composition function, $g°f$ is a surjective function.
(b) Prove or disprove: the composition function, $g°f$ is an injective function.

For each part, clearly state whether the assertion is true or not, followed by a proper justification.

(a) This claim is false. It's possible that $g°f$ is not surjective.

Let A = {1, 2, 3}, B = {4, 5} and C = {6, 7, 8}.
Let *f* = {(1,4), (2,4), (3, 5)} and *g* = {(4, 6), (5, 8)}.

For this particular example, $g°f$ = {(1,6), (2, 6), (3, 8)}. This function isn't surjective because there is no value, a, in the set A such that $g°f(a) = 7$, but $7 \in C$.

(b) This claim is also false. In the previously stated counter-example, the function $g°f$ is not injective because $g°f(1) = g°f(2) = 6$.

Grading, for both parts:

1 pt for clearly stating that the claim is false (for both parts), if this isn't clearly stated for a part, automatic 0/4 for the part.

2 pts for clearly specifying a valid counter-example with A, B, C, f, g and g(f(x)) fully written out. (Give 1 pt if the counter-example is invalid or incomplete)

1 pt for showing that the counter-example is just that.

4) (6 pts) Let r, s and t be roots of the cubic polynomial $f(x) = 3x^3 - 5x^2 + 7x - 9$. What is the cubic equation, with **<u>leading coefficient 9,</u>** with the roots $\frac{1}{r}$, $\frac{1}{s}$, and $\frac{1}{t}$?

Using the given information, we can set up the following equations that r, s and t satisfy:

$r + s + t = \frac{5}{3}$, $rs + rt + st = \frac{7}{3}$, $rst = \frac{9}{3}$

In order to get a cubic with the roots $\frac{1}{r}$, $\frac{1}{s}$, and $\frac{1}{t}$, we need to find the values of the following expressions:

$\frac{1}{r} + \frac{1}{s} + \frac{1}{t}$, $\frac{1}{rs} + \frac{1}{rt} + \frac{1}{st}$, and $\frac{1}{rst}$.

Let's solve for these:

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = \frac{st}{rst} + \frac{rt}{rst} + \frac{rs}{rst} = \frac{rs+rt+st}{rst} = \frac{\frac{7}{3}}{\frac{9}{3}} = \frac{7}{9}$$

$$\frac{1}{rs} + \frac{1}{rt} + \frac{1}{st} = \frac{t}{rst} + \frac{s}{rst} + \frac{r}{rst} = \frac{r+s+t}{rst} = \frac{\frac{5}{3}}{\frac{9}{3}} = \frac{5}{9}$$

$$\frac{1}{rst} = \frac{1}{\frac{9}{3}} = \frac{3}{9}$$

It follows that a cubic with these roots is $g(x) = x^3 - \frac{7}{9}x^2 + \frac{5}{9}x - \frac{3}{9}$. We can multiply this cubic through by any constant and not change its roots. Since the question requires a leading coefficient of 9, we multiply g(x) above by 9 to get the final answer: $\mathbf{9x^3 - 7x^2 + 5x - 3.}$

<u>$\mathbf{9x^3 - 7x^2 + 5x - 3}$</u>

Note: the pattern seen here that all the coefficients are in "reverse" order, following the sign switching rule is true in general. Just plug in $\frac{1}{x}$ for *x* in any arbitrary polynomial where we know 0 isn't a root (constant coefficient that is non-zero) and multiply through by x^n, where *n* is the degree of the polynomial and then simplify!

Grading: **1 pt writing down equations with r, s and t**
2 pts solving for 1/r + 1/s + 1/t
1 pt solving for 1/rs + 1/rt + 1/st
1 pt solving for 1/(rst)
1 pt mult by 9