<u>COT 3100 Sections 1/201 Exam #2 Solutions</u>

1) (6 pts) $n = 6^7 15^4$. How many divisors does n have?

$n = 6^7 15^4 = (2 \times 3)^7 (3 \times 5)^4 = 2^7 3^7 3^4 5^4 = 2^7 3^{11} 5^4$

This value is (7 + 1)(11 + 1)(4 + 1) = 8 x 12 x 5 = 40 x 12 = **480 divisors.**

<u>480</u>

Grading: 1 pt out of 6 if they answer 8 x 5 = 40
3 pts for breaking n down into prime factors (give partial as needed)
2 pts for plugging into the formula for # of divisors
1 pt for simplifying the product to 480

2) (7 pts) Although we didn't explain it in class, the prime sieve can work on a range of integers not starting with 1. The trick to making it work is figuring out the first number that you have to cross out in the range, since any one of the first p numbers in an arbitrary range of integers might be divisible by the prime p. Use this technique, determine all of the prime numbers in between 111 and 120. Note that because $120 < 121 = 11^2$, you only have to do four sets of cross offs for the four primes less than 11: 2, 3, 5 and 7. In your work below, write out each integer that is crossed off on each pass. (Note: The same integer might be crossed off more than once.) Then, write down the prime numbers in between 111 and 120, inclusive.

111 112 113 114 115 116 117 118 119 120

Cross Offs (more slots might be provided than necessary for each of the lists below.)

2: **<u>112</u>, <u>114</u>, <u>116</u>, <u>118</u>, <u>120</u>**

3: **<u>111</u>, <u>114</u>, <u>117</u>, <u>120</u>**

5: **<u>115</u>, <u>120</u>**

7: **<u>112</u>, <u>119</u>**

Primes in Range: **<u>113</u>**

Grading: 2 pts lists for 2 and 3, 1 pt lists for 5 and 7, 1 pt final answer (all items for 1 point must be perfectly correct to get the point, 1 pt may be awarded for the first two lists if they have small errors.)

3) (12 pts) Find all integer solutions (x, y) to the equation $305x + 110y = 35$.

Run GCD of 305 and 110:

305 = 2 x 110 + 85	61 = 2 x 22 + 17
110 = 1 x 85 + 25	22 = 1 x 17 + 5
85 = 3 x 25 + 10	17 = 3 x 5 + 2
25 = 2 x 10 + 5	5 = 2 x 2 + 1
10 = 2 x 5	

We can divide the whole equation through by 5 and equivalently find all solutions to

$61x + 22y = 7$

Rewrite the GCD statements above with 61 and 22 (done above in the right side). Then, run the Extended Euclidean Algorithm:

5 – 2 x 2 = 1
5 – 2(17 – 3 x 5) = 1
5 – 2 x 17 + 6 x 5 = 1
7 x 5 – 2 x 17 = 1
7(22 – 17) – 2 x 17 = 1
7 x 22 – 7 x 17 – 2 x 17 = 1
7 x 22 – 9 x 17 = 1
7 x 22 – 9(61 – 2 x 22) = 1
7 x 22 – 9 x 61 + 18 x 22 = 1
25 x 22 – 9 x 61 = 1

Multiply this equation through by 7:
(25 x 7) x 22 – (9 x 7) x 61 = 7
175 x 22 – 63 x 61 = 7

It follows that one solution to the given equation is $x = -63$, $y = 175$.

We can write all solutions as follows:

$\{ (x, y) \mid x = -63 + 22n, y = 175 - 61n, n \in Z \}$

We can recenter the solution (to look nicer by plugging n = 3) to get

$\{ (x, y) \mid x = 3 + 22n, y = -8 - 61n, n \in Z \}$

Grading: Getting GCD is 5 (or just dividing through by 5) = 2 pts
Extended Euclidean Algorithm to get to 25 x 22 – 63 x 61 = 1 or equiv = 6 pts
Multiplying through by 7 = 1 pt
Getting one solution – 1 pt, Adding offsets properly = 2 pts (signs = 1 pt)

4) (12 pts) Prove that for all non-negative integers, n, that $17 \mid (13^{2n} + 4^{2n+2})$.

Base case: n = 0, Plugging in n = 0, we get $13^{2(0)} + 4^{2(0)+2} = 13^0 + 4^2 = 1 + 16 = 17$
Since $17 = 1 \times 17$, we see that $17 \mid 17$ and the base case holds.

Inductive hypothesis: Assume for an arbitrarily chosen non-negative integer n = k that $17 \mid (13^{2k} + 4^{2k+2})$, meaning that there exists an integer c such that $13^{2k} + 4^{2k+2} = 17c$.

Inductive step: Prove for n = k + 1 that $17 \mid (13^{2(k+1)} + 4^{2(k+1)+2})$.

$$
\begin{aligned}
13^{2(k+1)} + 4^{2(k+1)+2} &= 13^{2k+2} + 4^{2k+2+2} \\
&= 13^{2k}(13^2) + 4^{2k+2}(4^2) \\
&= 169 \times 13^{2k} + 16 \times 4^{2k+2} \\
&= 153 \times 13^{2k} + 16 \times 13^{2k} + 16 \times 4^{2k+2} \\
&= 17 \times 9 \times 13^{2k} + 16\,(13^{2k} + 4^{2k+2}) \\
&= 17 \times 9 \times 13^{2k} + 16\,(17c)\text{, via the Inductive Hypothesis} \\
&= 17(9 \times 13^{2k} + 16c)
\end{aligned}
$$

Since c is an integer and k is a non-negative integer, it follows that $(9 \times 13^{2k} + 16c)$ is an integer and we've proven the inductive step.

We can conclude for all non-negative integers n, $17 \mid (13^{2n} + 4^{2n+2})$.

Alternate proof path: In the IH, solve for $13^{2k} = 17c - 4^{2k+2}$. Then in algebra, do this:

$$
\begin{aligned}
&= 169 \times 13^{2k} + 16 \times 4^{2k+2} \\
&= 169 \times (17c - 4^{2k+2}) + 16 \times 4^{2k+2} \\
&= 169(17c) - 169 \times 4^{2k+2} + 16 \times 4^{2k+2} \\
&= 169(17c) - 153 \times 4^{2k+2} \\
&= 17(169c - 9 \times 4^{2k+2})
\end{aligned}
$$

There are other equivalents.

Grading: Base case = 1 pt, Stating IH = 1 pt, Stating IS = 1 pt
Correctly splitting off 13^2 and 4^2 = 2 pts
Isolating IH one which way or the other = 3 pts
Plugging in IH correctly = 2 pts
Finishing it up = 2 pts

5) (12 pts) Let F_n denote the n^{th} Fibonacci number. (Recall $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for all integers n > 1.) Prove that the following summation holds for all positive integers n:

$$\sum_{i=1}^{n} (-1)^{i-1} F_i = (-1)^{n-1} F_{n-1} + 1$$

Base case: n = 1, LHS = $\sum_{i=1}^{1} (-1)^{i-1} F_i = (-1)^{1-1} F_1 = 1$

RHS = $(-1)^{1-1} F_{1-1} + 1 = F_0 + 1 = 0 + 1 = 1$

Thus, the given formula is true for n = 1 and the base case holds.

Inductive hypothesis. Assume for an arbitrarily chosen positive integer n = k that

$$\sum_{i=1}^{k} (-1)^{i-1} F_i = (-1)^{k-1} F_{k-1} + 1$$

Inductive step: Prove for n = k+1 that

$$\sum_{i=1}^{k+1} (-1)^{i-1} F_i = (-1)^{k} F_{k} + 1$$

$$\sum_{i=1}^{k+1} (-1)^{i-1} F_i = \left[\sum_{i=1}^{k} (-1)^{i-1} F_i\right] + (-1)^{k} F_{k+1}$$

$$= (-1)^{k-1} F_{k-1} + 1 + (-1)^{k} F_{k+1}, \text{ via IH}$$

$$= (-1)^{k}(-F_{k-1} + F_{k+1})$$

$$= (-1)^{k} F_{k}\ , \text{ via Fibonacci recurrence}$$

Grading: Base case = 1 pt, Stating IH = 1 pt, Stating IS = 1 pt
Correctly splitting off last term = 2 pts
Plugging in IH = 2 pts
Factoring out $(-1)^k$ = 2 pts
Using Fib recurrence to finish = 3 pts

6) (8 pts) The arithmetic sequence $a_1, a_2, \ldots, a_{100}$ has a common difference of 2 and a sum of 350. What is the sum $\sum_{i=1}^{50} a_{2i-1}$ equal to? (Note: This problem can be solved without determining a_1.)

Let S = $\sum_{i=1}^{50} a_{2i-1}$. Let T = $\sum_{i=1}^{50} a_{2i} = \sum_{i=1}^{50}(a_{2i-1} + 2) = \left(\sum_{i=1}^{50} a_{2i-1}\right) + \sum_{i=1}^{50} 2 = S + 100$

Finally, note that the given information tells us that S + T = 350. Since T = S + 100, we have:

S + S + 100 = 350
2S = 250
<u>S = 125</u>

Grading: Noticing that even terms sum to 100 more than odd terms = 4 pts
Setting up equation with sum of even and odd terms = 2 pts
Arriving at the final answer from the equation = 2 pts

7) (6 pts) N and M are positive integers such that $160N = M^4$. What is the minimum possible value of N?

$160N = M^4$
$2^5(5)N = M^4$

At a minimum, $5 \mid M$ and $2^2 \mid M$ because the RHS has more than 4 copies of 2, thus the only way to satisfy the equation is if M is divisible by at least 4. Minimizing M also helps minimize N, so plug this in:

$2^5(5)N = 4^4 5^4$
$2^5(5)N = 2^{2(4)} 5^4$
$2^5(5)N = 2^8 5^4$
$N = 2^3 5^3 = (2 \times 5)^3 =$ **<u>1000</u>**

Grading: Prime Factorizing 160 = 2 pts
Determining that M = 20 for best answer = 2 pts
Plugging in and solving for the corresponding N correctly = 2 pts
<u>Guess and Check System gets automatic 2 out of 6</u>

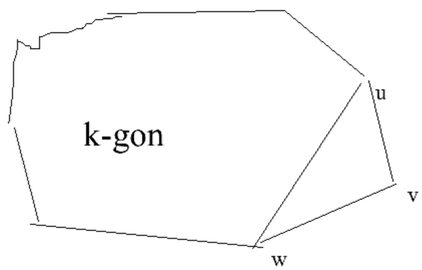
8) (10 pts) The sum of the interior angles of a convex n-gon is 180(n-2) degrees. Under the assumption that the sum of the degrees in a triangle is 180 degrees, prove this result via induction on n for all integers $n \geq 3$.

Base case: n = 3, A triangle has its sum of degrees equal to 180 (given).
Plugging into the formula it should have (3 – 2)180 = 180 degrees.
The given formula is true for n = 3.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k, with $k \geq 3$, that the sum of the degrees of the interior angles of a convex k-gon is 180(k – 2) degrees.

Inductive step: Prove for n = k + 1, that the sum of the degrees of the interior angles of a convex (k+1)-gon is 180(k + 1 – 2) = 180(k – 1) degrees.

Proof of inductive step: Take an arbitrary convex (k+1)-gon and pick a vertex. Call this vertex v. Find the vertex that is adjacent to v in the clockwise direction and call this w and find the vertex that is adjacent to v in the counter-clockwise direction and call this u. Draw a line segment between u and w as follows:



When we do this, we partition the (k+1)-gon into a k-gon and a triangle, uvw. (Note: This works because the k+1 gon is convex.) (Note: I've intentionally drawn the top left "fuzzy" simply indicating k+1 total sides instead of a hard-coded number, so that corner may have several sides...)

Using the inductive hypothesis, the sum of the angles in the k-gon is 180(k – 2). Our k+1 gon has all of these angles plus the angles shown in the triangle, which sum to 180 degrees. (Note that the k+1 gon has two angles that are bigger than the k-gon, we equal these out by adding angles vuw and vwu.) Adding these two together we get 180(k – 2) + 180 = 180(k – 2 + 1) = 180(k – 1), proving the inductive step.

It follows that for all positive integers n with $n \geq 3$, a convex n-gon has a sum of interior angles equal to 180(n – 2).

Grading: Base case = 1 pt, Stating IH = 1 pt, Stating IS = 1 pt
Splitting off Triangle Clearly = 3 pts
Using IH = 2 pts, Adding Triangle Angle = 1 pt
Finishing up the addition = 1 pt

9) (2 pts) In the song Back to December, the event discussed occurred in which month?
December **(Grading: Give to All)**