COT 3100 Recitation: Roots of Polynomials Solutions

<u>Solutions to Set #1 Problems</u>

1) Compute the sum of all the roots of $(2x + 3)(x - 4) + (2x + 3)(x - 6) = 0$.

<u>Solution</u>
Factor to get:

$(2x + 3)(x – 4 + x – 6) = 0$
$(2x + 3)(2x – 10) = 0$

Roots are -3/2 and 5, respectively. Their sum is **<u>7/2.</u>**

2) Compute the following product $(3x^3 + 2x - 4)(2x^4 - 5x^2 + x - 6)$.

<u>Solution</u>

$$
\begin{array}{rl}
(3x^3 + 2x - 4)(2x^4 - 5x^2 + x - 6) = & 6x^7 - 15x^5 + 3x^4 - 18x^3 \\
& \quad + 4x^5 \qquad - 10x^3 + 2x^2 - 12x \\
& \qquad -8x^4 \qquad + 20x^2 - 4x + 24 \\
\hline
& \mathbf{6x^7 - 11x^5 - 5x^4 - 28x^3 + 22x^2 - 16x + 24}
\end{array}
$$

3) The r and s be the roots of the quadratic equation $x^2 + 3x - 6 = 0$. What is the quadratic equation with leading coefficient 1 that has roots $\frac{r}{s}$ and $\frac{s}{r}$?

<u>Solution</u>
Using the given information, we know that $r + s = -3$ and $rs = -6$. Now let's solve for:

$$\frac{r}{s} + \frac{s}{r} = \frac{r^2 + 2rs + s^2 - 2rs}{rs} = \frac{(r+s)^2 - 2rs}{rs} = \frac{(-3)^2}{-6} - 2 = -\mathbf{\frac{7}{2}}$$

4) If x, y and z are positive real numbers satisfying $x + \frac{1}{y} = 4$, $y + \frac{1}{z} = 1$, and $z + \frac{1}{x} = \frac{7}{3}$, then what is the value of xyz?

<u>Solution</u>
Adding the three given equations tells us:

$$x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 4 + 1 + \frac{7}{3} = \frac{22}{3}$$

Multiplying the three equations tells us:

$$\left(x+\frac{1}{y}\right)\left(y+\frac{1}{z}\right)\left(z+\frac{1}{x}\right)=\frac{28}{3}$$

$$xyz+x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{xyz}=\frac{28}{3}$$

Substituting for the middle six terms from the previous equation, we get:

$$xyz+\frac{22}{3}+\frac{1}{xyz}=\frac{28}{3}$$

$$xyz+\frac{1}{xyz}=2$$

Let S = xyz, so we now aim to find S. Substitute:

$$S+\frac{1}{S}=2$$

$$S^2+1=2S$$

$$S^2-2S+1=0$$

$$(S-1)^2=0$$

It follows that S = xyz = **1.**

Solutions to Set #2 Problems

1) Compute the sum of all of the roots in the equation $2x^2 - 5x + 7 = 0$.

Solution
The sum of the roots of the quadratic $ax^2 + bx + c = 0$ is $-b/a$. Thus, for this equation the sum of the roots is $-\left(-\frac{5}{2}\right) = \frac{5}{2}$.

2) Compute the following product $(2x^3 - 7x^2 + 3x - 1)(4x^2 + 2x - 5)$

Solution

$$(2x^3 - 7x^2 + 3x - 1)(4x^2 + 2x - 5) = \begin{array}{l}(8x^5 + 4x^4 - 10x^3) + \\ (-28x^4 - 14x^3 + 35x^2) + \\ (12x^3 + 6x^2 - 15x) + \\ (-4x^2 - 2x + 5)\end{array}$$

$= \mathbf{8x^5 - 24x^4 - 12x^3 + 37x^2 - 17x + 5}$

3) The r and s be the roots of the quadratic equation $x^2 + 7x - 11 = 0$. What is the quadratic equation with leading coefficient 1 that has roots r^2 and s^2?

Solution
If an equation has roots r^2 and s^2, then the sum of those roots is $r^2 + s^2$ and the product of those roots is r^2s^2. If we can find the values of these two expressions, we can recreate the desired quadratic.

Using the given information, we see that:

$r + s = -7$
$rs = -11$

So, take the first equation and square it and simplify:

$(-7)^2 = (r + s)^2 = r^2 + 2rs + s^2 = r^2 + 2(-11) + s^2$

Now, solve for $r^2 + s^2$:

$49 = r^2 - 22 + s^2$
$71 = r^2 + s^2$

Since $rs = -11$, it follows that $(rs)^2 = r^2s^2 = (-11)^2 = 121$.

Thus, the desired quadratic is $\mathbf{x^2 - 71x + 121 = 0}$.

4) If $x+\frac{1}{x}=7$, what is $x^3+\frac{1}{x^3}$?

Solution

Cube the given expression:

$$(x+\frac{1}{x})^3 = 7^3$$

$$x^3+3x+\frac{3}{x}+\frac{1}{x^3}=343$$

$$x^3+3(x+\frac{1}{x})+\frac{1}{x^3}=343$$

But we know what $x+\frac{1}{x}$ is, so let's substitute for that:

$$x^3+3(7)+\frac{1}{x^3}=343$$

$$x^3+\frac{1}{x^3}=\mathbf{322}$$