COT 3100H Final Exam - Part A (Recitation Topics) Solutions

1) (10 pts) A course grade is comprised of 5 Quizzes, 10 Homework Assignments and 4 Exams. The quizzes are worth 30% of the course grade, the homework assignments are worth 10% of the course grade and the exams are worth the remaining 60% of the course grade. Each of the quizzes is out of 60 points, each of the homework assignments is out of 20 points and each of the exams is out of 100 points. Each quiz is worth the same amount of the course grade, each homework assignment is worth the same amount of the course grade and each exam is worth the same amount of the course grade. Note that a quiz, homework assignment and exam are each worth different amounts of the course grade.

Thus far in the semester, 3 quizzes have been administered, 7 homework assignments have been given and 1 exam has been given.

Let **X** represent the current grade (as a percentage) reported in Webcourses (which isn't accurate), and
Let **Y** represent the actual current grade (as a percentage) of the student.

With proof, determine the **maximum** possible value of X - Y. (Note: a "greedy" strategy without any Calculus is sufficient for the maximization.) What percentages in each category achieve this maximum difference? (Note: there are no absolute value bars around X - Y.)

Solution

The grade that webcourses reports is always based on the final end of term relative weights. Let Q represent a student's current average percentage for the 3 quizzes, H represent the current average percentage for 7 homework assignments and E represent a student's current average for one exam. According to Webcourses, that student's percentage would be:

$X = .3Q + .1H + .6E$

Unfortunately, the quizzes ACTUALLY represent $\frac{3}{5}(30\%) = 18\%$ of the total course grade.
The homework ACTUALLY represents $\frac{7}{10}(10\%) = 7\%$ of the total course grade.
The exam ACTUALLY represents $\frac{1}{4}(60\%) = 15\%$ of the total course grade.
Thus, only 18% + 7% + 15% = 40% of the course grade has been accounted for, so far. Using these numbers, we see that as of right now in the course, the relative weights of each category ACTUALLY are:

Quizzes are currently worth $\frac{18\%}{40\%} = 45\%$, of the current grade. Homework is currently worth $\frac{7\%}{40\%} = 17.5\%$, of the current grade. The Exam is currently worth $\frac{15\%}{40\%} = 37.5\%$, of the current grade.

Thus a student's current grade in the course can be represented as: $Y = .45Q + .175H + .375E$

It follows that:

$X - Y = .3Q + .1H + .6E - (.45Q + .175H + .375E) = .225E - .15Q - .075H$

If we want to maximize this quantity with the constraint that $0 \le E, Q, H \le 100$ (while not explicitly stated, common sense dictates this restriction), then it's clear we should set E = 100, Q, H = 0. This is because any added exam score adds to this quantity and any added quiz or homework score subtracts from it. With these settings, we have

X - Y = .225(100) - .15(0) - .075H = **22.5%**

Thus, while unlikely, in this scenario (where a student gets a perfect score on one exam, but gets 0s for everything else), Webcourses would report a grade that is 22.5% higher than the student's actual grade.

Grading: 2 pts for WC expression, 6 pts for Actual expression
2 pts for logic of how to maximize and the final answer.

2) (10 pts) Let A be an arithmetic sequence of 30 terms, with a first term of 2 with a common difference of 3. Let B be an infinite geometric sequence with the same first term as A AND the same sum as A. What is the common ratio of B? **Express your answer as a fraction in lowest terms. Feel free at the very end to multiply two numbers on your calculator to speed up computation, but the rest of the work must be done by hand.**

Solution

Let the terms of a be $a_1, a_2, \ldots, a_{30}$. Using the given information, we have $a_{30} = 2 + (30-1)*3 = 89$. It follows that the sum of the sequence is $S_{30} = \frac{(a_1+a_{30})30}{2} = (2+89)15 = 91 \times 15$.

Let r be the common ratio of the sequence B and let S be the sum of this sequence. Then, we have

$$S = \frac{2}{1-r} = 91 \times 15$$

$$1 - r = \frac{2}{91 \times 15}$$

$$r = 1 - \frac{2}{91 \times 15} = \frac{1365 - 2}{1365} = \mathbf{\frac{1363}{1365}}$$

Grading: 2 pts a_{30}, 3 pts sum of A, 2 pts set up eqn for r, 3 pts solve for r as fraction.

3) (10 pts) Determine the following sum, simplifying your result to a single value without the use of a calculator. (Note: $2^{10} = 1024$.)

$$\sum_{i=1}^{100} log_{1024}(4^i)$$

Solution

$$\sum_{i=1}^{100} log_{1024}(4^i) = \sum_{i=1}^{100} i(log_{1024}4) = \sum_{i=1}^{100} i\left(\frac{1}{5}\right) = \frac{100 \times 101}{2 \times 5} = \mathbf{1010}$$

Grading: Lots of ways to do this…Here is a breakdown of grading for the method above:

2 pts - bring i down
4 pts - simplify $log_{1024}4$ correctly
3 pts - sub in formula for sum of i
1 pt - get to correct final answer

COT 3100H Final Exam - Part B (Logic, Sets, Relations, Functions) - 30 pts Solutions

1) (6 pts) Prove or disprove the following statement over the universe of positive real numbers:

$$\forall x \exists y[3(x - y) = 2xy + 1]$$

Solution

This statement is false, but it takes some careful analysis to get to this conclusion. This analysis follows.

We must show that for all x, there exists a value y. Thus, we must solve for a value of y that makes the statement true, if possible, or so what doing so is not possible by finding a single value of x, for which there is no y. Let's just try out some algebra at first:

$$3(x - y) = 2xy + 1$$
$$3x - 3y = 2xy + 1$$
$$2xy + 3y = 3x - 1$$
$$y(2x + 3) = 3x - 1$$

At this juncture, it's important to note that x > 0 thus, 2x +3 > 0 (most importantly, it's not 0), so we are allowed to divide this equation by 2x + 3:

$$y = \frac{3x - 1}{2x + 3}$$

Thus, we have an expression for y which makes the statement true for any given x. Unfortunately, we see that our domain is the set of positive reals and that it's possible that for some positive real values of x, the corresponding value of y that makes it true is not positive. Thus, the given statement is NOT true. With our work, we can now present a counter-example:

Let x = $\frac{1}{6}$. For this positive value of x, there is only one value of y that makes the given statement true and that value is $y = \frac{3(1/6)-1}{2(1/6)+3} = \frac{-1/2}{10/3} = -\frac{3}{20}$. But, unfortunately, this value is not positive. Thus, for x = $\frac{1}{6}$, we have proven that there is NO positive real number y which makes the statement true. It follows that the statement is NOT true for the domain of all positive real numbers x and y.

Grading: 1 pt for no answer. 4 pts for solving for y. 1 pt for final answer. (Alternatively, full credit for any solution that just gives a counter-example off the bat. 5 out of 6 for any solution that properly solves for y and says that the claim is true without realizing the effect of the positive real restriction.)

2) (6 pts) Let A and B be sets such that $|A - B| = 2$, $|B - A| = 3$ and $|A \cap B| = 4$. Determine the value of $|\wp(A \cup B)|$. Recall that $\wp(X)$ denotes the power set of the set X.

Solution

Recall that the sets A - B, B - A and $A \cap B$, are all mutually exclusive, by definition, and that

$$A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$$

This can be verified with a membership table with four rows. It follows based on the counting principle of addition that:

$$|A \cup B| = |A - B| \cup |B - A| \cup |A \cap B|$$

Thus, we have:

$$|A \cup B| = 2 + 3 + 4 = 9$$

Finally, for a power set of n elements has 2^n elements itself. It follows that

$$|\wp(A \cup B)| = \mathbf{2^9}$$

Grading: 4 pts for the logic to arrive at the fact that there are 9 items in A union B (4 pts is for the logic not the result), 2 pts to get to the final answer.

3) (10 pts) Define the following relation over the set of positive integers:

R = { (a, b) | a + b is a prime number }

With proof determine if R is (a) reflexive, (b) irreflexive, (c) symmetric, (d) anti-symmetric and € transitive. For each answer (yes/no) provide appropriate justification.

Solution

(a) No, (3, 3) is NOT in R because 3 + 3 = 6 and 6 is NOT prime.

(b) No, (1, 1) IS in R because 1 + 1 = 2 and 2 is prime.

(c) Yes, for all (a, b) in R, we must have that a + b is prime, but this means that b + a = a + b, and that b + a is prime. It follows that (b, a) is in R, as desired and R is symmetric.

(d) No, (2, 3) is in R and (3, 2) is in R but 2 isn't equal to 3. (2 + 3 = 5 and 5 is prime.)

(e) No, (2, 3) is in R, (3, 2) is in R, but (2, 2) is NOT in R because 2 + 3 = 3 + 2 = 5 which is prime, but 2 + 2 = 4, and 4 is not prime.

Grading: 1 pt for each answer, 1 pt for each reason

4) (8 pts) Let a function f: A → B be an injective function and g: B → C be a surjective function. With proof, determine if (a) g°f must be an injective function, and (b) g°f must be a surjective function. (If the answer is yes for a part, prove it in the general case. If the answer is no for a part, give a specific counter example where you completely specify A, B, C, f and g and explain how your specific example is a counter-example.)

<u>Solution</u>

The answer to both parts is no. A single counter-example can be created to show that neither conclusion is correct.

Let

A = {1,2}

B = {a,b,c}

C = {x, y}

f = {(1,a), (2, b)}

g = {(a, x), (b, x), (c, y)}

In this example, we have:

g°f = {(1,x), (2, x)}

This function is not injective because there are two different values, 1 and 2, such that g(f(1)) = g(f(2)). This function is not surjective because there is no value a, in the set A such that g(f(a)) = y.

The key here is that since there is no requirement for g to be injective, we can get multiple values in B to map to the same value in C via the function g. Then, in this way, we can make an example where the function composition is not injective. Secondly, to make the function composition not surjective, since f doesn't need to be surjective, we can simply just have f not map to a value in B that is "critical" in making the composition function surjective.

Grading: 2 pts for saying no to both parts, 3 pts for a counter example that doesn't work, 6 pts for a counter example that does work, max 2 pts for any response that says that both parts are true. For any response that says one part is true and one is false, max pts 4 for the false part and 1 pt for the true part.

COT 3100H Final Exam - Part C (Number Theory, Induction) - 30 pts Solution

1) (10 pts) Determine all integer solutions to the equation 504x + 150y = 24.

Solution

Run the Euclidean Algorithm to find gcd(504, 150):

504 = 3 x 150 + 54
150 = 2 x 54 + 42
54 = 1 x 42 + 12
42 = 3 x 12 + 6
12 = 2 x 6

Since 6 | 24, there are integer solutions to the given equation.

Now, run the Extended Euclidean Algorithm:

42 - 3 x 12 = 6
42 - 3(54 - 42) = 6
42 - 3 x 54 + 3 x 42 = 6
4 x 42 - 3 x 54 = 6
4(150 - 2 x54) - 3 x 54 = 6
4 x 150 - 8 x 54 - 3 x 54 = 6
4 x 150 - 11 x 54 = 6
4 x 150 - 11(504 - 3 x 150) = 6
4 x 150 - 11 x 504 + 33 x 150 = 6
37 x 150 - 11 x 504 = 6

Now, multiply this equation through by 4 since 6 x 4 = 24:

(37 x 4) x 150 - (11 x 4) x 504 = 6 x 4
148 x 150 - 44 x 504 = 25

Thus, one solution for x and y that satisfies the given equation is x = - 44, y = 148.

To get all solutions, notice that we can simultaneously add $\frac{150}{6} = 25$ to any solution for x and subtract $\frac{504}{6} = 84$ from the corresponding solution for y, since 504 x 25 - 150 x 84 = 0 to produce another solution. It follows that the total solution set is as follows:

{ (x, y) | x = -44 + 25c, y = 148 - 84c, $c \in Z$ }

Grading: 2 pts Euclidean
5 pts Extended
1 pt one solution
2 pts all solutions

2) (4 pts) Let k equal the maximum integer such that 500! is divisible by 40^k. What is k?

Solution

40 = 8 x 5 = 2^3 x 5. Thus, for an integer to be divisible by 40^k, it must be divisible by 2^{3k} and 5^k. Let's determine how many times 2 and 5 divide evenly into 500! using the algorithm given in class:

2 | 500
2 | 250
2 | 125
2 | 62 R1
2 | 31
2 | 15 R 1
2 | 7 R 1
2 | 3 R 1
2 | 1 R 1

Thus, 2 divides evenly into 500! a total of 250 + 125 + 62 + 31 + 15 + 7 + 3 + 1 = 494 times. Thus, the maximum value of k for which 2^{3k} divides evenly into 500! Is $\left\lfloor \frac{494}{3} \right\rfloor = 164$ times.

Repeat for 5:

5 | 500
5 | 100
5 | 20
5 | 4

The maximum value of k for which 5^5 divides evenly into 500! is 100 + 20 + 4 = 124 times. It follows that the limiting factor is 5 and that the maximum value of k for which 40^k divides evenly into 500! is min(164, 124) = **124.**

Grading: 1 pt for how many times 2 divides in 1, pt for how many times 8 divides in, 1 pt for how many times 5 divides in, 1 pt for final answer.

3) (8 pts) Define a recurrence relation as follows:

$a_0 = 1$, $a_1 = 3$
$a_n = 7a_{n-1} - 12a_{n-2}$, for all integers $n \geq 2$.

Make a guess for a closed form formula for a_n in terms of n and prove that guess via strong induction on n with 2 base cases. (Note: a guess would be something like $a_n = 2n^2 + 1$. Of course, this is not the correct answer! Just trying to show that the guess is to be some formula in terms of n which correctly describes the value of a_n recursively defined above.)

Solution
Let's list out a couple more terms:

$a_2 = 7a_1 - 12a_0 = 7(3) - 12(1) = 9$
$a_3 = 7a_2 - 12a_1 = 7(9) - 12(3) = 27$

The sequence starts, 1, 3, 9 and 27. These look like powers of 3. Thus our guess is:

For all non-negative integers n, $a_n = 3^n$. We will prove this claim via strong induction on n.

Base case: $n = 0$ LHS $= a_0 = 1$, RHS $= 3^0 = 1$, these are equal
$n = 1$ LHS $= a_1 = 3$, RHS $= 3^1 = 3$, these are equal
The base case holds for $n = 0, 1$

Inductive hypothesis: Assume for all non-negative integers $n \leq k$, where k is an integer greater than or equal to 1 that $a_n = 3^n$.

Inductive step: Prove for $n = k+1$ that $a_{k+1} = 3^{k+1}$.

$a_{k+1} = 7a_k - 12a_{k-1}$
$= 7(3^k) - 12(3^{k-1})$, via the strong inductive hypothesis
$= 7(3)(3^{k-1}) - 12(3^{k-1})$
$= 21(3^{k-1}) - 12(3^{k-1})$
$= 9(3^{k-1})$
$= 3^2(3^{k-1})$
$= 3^{k+1}$

This completes the proof of the inductive step. We can conclude that for all non-negative integers n, that $a_n = 3^n$.

Grading: 2 pts guess
1 pt BC
1 pt IH
1 pt IS
3 pts proof
Can only obtain 1 pt max if guess is wrong.

4) (8 pts) Prove for all non-negative integers n that $7 \mid (5^{2n+1} + 2^{2n+1})$

Solution

Use induction on n.

Base case: $n = 0$, $5^{2(0)+1} + 2^{2(0)+1} = 5^1 + 2^1 = 7$. Since $7 \mid 7$, it follows that the base case is true.

Inductive hypothesis: Assume for an arbitrarily chosen non-negative integer $n = k$ that $7 \mid (5^{2k+1} + 2^{2k+1})$.

Inductive step: Prove for $n = k+1$ that $7 \mid (5^{2(k+1)+1} + 2^{2(k+1)+1})$. Namely, prove that $7 \mid (5^{2k+3} + 2^{2k+3})$

$$5^{2k+3} + 2^{2k+3} = 5^2 5^{2k+1} + 2^2 2^{2k+1}$$

$$= 25(5^{2k+1}) + 4(2^{2k+1})$$

$$= 21(5^{2k+1}) + 4(5^{2k+1}) + 4(2^{2k+1})$$

$$= 21(5^{2k+1}) + 4[5^{2k+1} + 2^{2k+1}]$$

$= 21(5^{2k+1}) + 4[7c]$, for some integer c, via the inductive hypothesis since $7 \mid (5^{2k+1} + 2^{2k+1})$.

$$= 7\,[3(5^{2k+1}) + 4c]$$

Since c and k are non-negative integers, it follows that $3(5^{2k+1}) + 4c$ is also an integer. Thus, the inductive step is proven to be true.

It follows that the given claim is true for all non-negative integers n.

Grading: BC - 1 pt
IH - 1 pt
IS - 1 pt
Factor out - 1 pt
Split term - 1 pt
Regroup - 1 pt
Use IH - 1 pt
Factor out 7 and complete - 1 pt

COT 3100H Final Exam - Part D (Counting, Probability) - 35 pts Solution

1) (5 pts) A class has 10 girls and 15 boys. A quiz bowl team of 5 students from the class will be chosen such that at least 2 girls are on the team. In how many ways can the team be selected? Please leave your answer as a sum/difference/product of combinations.

Solution

The total # of ways the team can be selected is in $\binom{10+15}{5} = \binom{25}{5}$ ways.

Of these, here are the ways to select teams that consist of 0 and 1 girl, respectively:

$\binom{10}{0}\binom{15}{5}$ and $\binom{10}{1}\binom{15}{4}$.

It follows that the total number of valid team constructions is:

$$\binom{25}{5} - \binom{10}{0}\binom{15}{5} - \binom{10}{1}\binom{15}{4}$$

Grading: 1 pt for total, 2 pts for each subtracted term. If 4 terms added, 1 pt per added term 1 bonus point for getting it correct.

2) (10 pts) A landscape architect intends on buying 50 plants. She will buy some hibiscus, bougainvillea, azalea, hydrangea and lilac. She will buy in between 5 and 20 of each variety. In how many different ways can she buy the plants?

Solution

First, have her buy 5 of each variety so that there are 25 plants left to buy. Now, the new restriction is that she has to buy at most 15 of each type of plant. Let a, b, c, d and e represent the number of hibiscus, bougainvillea, azalea, hydrangea and lilac that she buys (after the initial 25 plants.) Thus, we want the number of non-negative integer solutions to:

$$a + b + c + d + e = 25$$

where a, b, c, d, e ≤ 15. The total number of solutions, using the formula for combinations with repetition is $\binom{25+5-1}{5-1} = \binom{29}{4}$.

From these, we must subtract out solutions where a > 15 or b > 15 or c > 15 or d > 15 or e > 15. First observe that it's impossible for any two of these inequalities to be satisfied because if they were, the sum of the five variables would be at least 32. Secondly, notice that the number of solutions where a > 15 equals the number where b > 15, since there is no difference between and b in the equation above. It follows that if we can find the # of solutions with a > 15, then we have the number of solutions for the other four inequalities.

To solve for the number of ways to satisfy the equation with a > 15, we will go ahead and buy 16 more hibiscus. Thus, we are looking for the number of ways to solve the equation

$$a' + b + c + d + e = 9$$

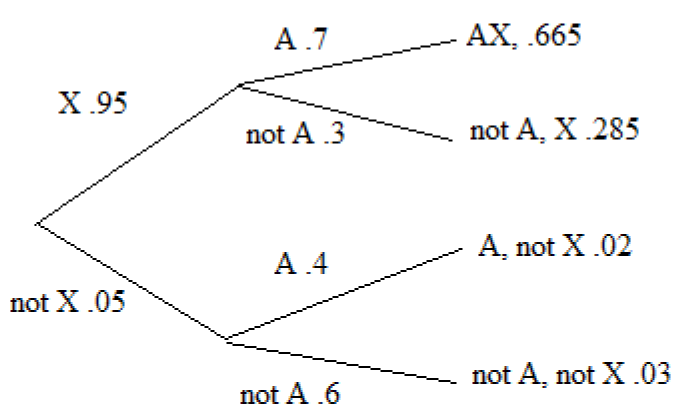
where a' represents the extra # of hibiscus past the 16. This equation has $\binom{9+5-1}{5-1} = \binom{13}{4}$ solutions.

It follows that the final answer to the question is $\boldsymbol{\binom{29}{4} - 5\binom{13}{4}}$.

Grading: 4 pts total # of answers
4 pts # of solutions with one variable above 20
1 pt multiply by 5
1 pt subtract

3) (5 pts) Jessalyn has a lucky pair of shoes she wears to exams. Given that she is wearing her shoes, she gets an A on an exam 70% of the time. If she is not wearing her lucky pair of shoes, she gets an A only 40% of the time. Luckily she remembers to wear her lucky pair of shoes 95% of the time. On a particular exam, Jessalyn did not get an A. What is the probability she was wearing her lucky shoes on that day? **Express your answer as a fraction in lowest terms.**

Solution
Here is a probability tree showing the situation. Let X represent the event that she is wearing her shoes and the event A represent that she gets an A on an exam.



P(not A) = .285 + .03 = .315, P(X | notA) = $\frac{P(X \cap \bar{A})}{P(\bar{A})} = \frac{.285}{.315} = \frac{285}{315} = \boldsymbol{\frac{19}{21}}$

Grading: 2 pts tree or equivalent, 1 pt numerator, 1 pt denominator, 1 pt fraction in lowest terms

4) (10 pts) Define a continuous random variable X as follows:

$$X = kx^2, 0 \le x \le 3$$
$$= 0, \text{otherwise}$$

for a constant k.

(a) Find k.
(b) Find E(X) (expectation of X).
(c) Find Var(X). (Note: Var(X) = $E(X^2)$ - $[E(X)]^2$. Also, translate the definition of E(X) and $E(X^2)$ from discrete random variables to continuous random variables. Namely, you are multiplying each probability by x for the first definition and by x^2 for the second and adding.)

Solution

(a) $\int_0^3 kx^2 dx = 1$

$\frac{k}{3}x^3|_0^3 = 9k = 1$

$k = \mathbf{\frac{1}{9}}$

(b) E(X) = $\int_0^3 x(\frac{1}{9}x^2)dx = \int_0^3 \frac{1}{9}x^3 dx = \frac{1}{36}x^4|_0^3 = \frac{81}{36} = \mathbf{\frac{9}{4}}$

(c) $E(X^2)$ = $\int_0^3 x^2(\frac{1}{9}x^2)dx = \int_0^3 \frac{1}{9}x^4 dx = \frac{1}{45}x^5|_0^3 = \frac{243}{45} = \frac{27}{5}$

Var(X) = $E(X^2)$ - $[E(X)]^2$ = $\frac{27}{5} - \left(\frac{9}{4}\right)^2 = \frac{27}{5} - \frac{81}{16} = \frac{27(16)-81(5)}{80} = \mathbf{\frac{27}{80}}$

Grading: 3 pts k, 3 pts E(X), 4 pts Var(X)

5) (5 pts) In the game of Connect Four, how many of your pieces do you have to place in a row to win?

Four (Grading: 5 pts give to all)