COT 3100 Recitation: Probability 1 Solutions

Set #1

1) Disease A occurs in 0.1% of the population. If a person does NOT have the disease and takes a test for the disease, the test correctly indicates that they don't have the disease 96% of the time. If a person has the disease and takes the same test, the test correctly indicates that they do have the disease 98% of the time. Given that you've taken the test and have tested positive for disease A, what is the probability you actually have disease A? Given that you've taken the test and have tested negative for disease A, what is the chance that you have the disease anyway?

Solution

Let $A = has\ disease\ A$ and $B = tested\ positive\ for\ A$

$\boldsymbol{a})\ P(A|B) = \dfrac{P(B\ |A)P(A)}{P(B)}$

$P(B) = P(B|A)P(A) + P(B|A)P(A)\ = .98 * .001\ + .04 * .999\ = 0.04094$

$$\frac{P(B\ |A)P(A)}{P(B)} = \frac{.98 * 0.001}{.04094} = 0.024 = \mathbf{2.4}\%$$

$b)\ P(A|\bar{B}) = \dfrac{P(\bar{B}\ |A)P(A)}{P(\bar{B})}$

$P(\bar{B}) = P(\bar{B}|A)P(A) + P(\bar{B}|\bar{A})P(\bar{A})\ = .02 * .001 + .96 * .999\ = 0.959$

$$\frac{P(\bar{B}\ |A)P(A)}{P(\bar{B})} = \frac{0.2 *\ .001}{.959} = 0.000021 = \mathbf{0.0021}\%$$

2) Anderson gets 80% of the multiple choice questions in COT 3100 he answers. Given a set of 15 questions, what is the chance that he gets at least 13 of them? Write your answer using combinations and then use a calculator to get a decimal approximation for it.

Solution

Using the binomial probability theorem, the probability of getting exactly k questions right out of n when the probability of getting a particular question is p (and the probability of getting each question is independent of the others) is $\binom{n}{k}p^k(1-p)^{n-k}$. Thus, we can just sum over k = 13, 14 and 15 to obtain:

$$P(at\ least\ 13\ questions\ correct) = \binom{15}{13}(.8)^{13}(.2)^2 + \binom{15}{14}(.8)^{14}(.2)^1 + \binom{15}{15}(.8)^{15}$$
$$= \mathbf{.398}$$

3) If A and B are events and p(A) = 4/9, p(A ∩ B) = 2/5, p(A | B) = 1/2 calculate p(B), p(B|A) and p(B |$\bar{A}$), are A and B independent? Mutually exclusive?

Solution

$$P(B) = \frac{P(A \cap B)}{P(A|B)} = \frac{2/5}{1/2} = \mathbf{4/5}$$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} = \frac{\frac{1}{2} * \frac{4}{5}}{\frac{4}{9}} = \mathbf{9/10}$$

$$P(B|\bar{A}) = \frac{P(\bar{A}|B)P(B)}{P(\bar{A})} = \frac{\frac{1}{2} * \frac{4}{5}}{\frac{5}{9}} = \mathbf{18/25}$$

A and B are not independent, since $P(B|A) \neq P(B)$
A and B are not mutually exclusive, since $P(A \cap B) \neq 0$

4) Bob has been chosen for a half-time promotion at a local basketball game. He will take as many three point shots as he can in 60 seconds. If he makes at least two of the shots, he wins $1000. After some practice, Bob knows that he'll get off 9 shots in the given time and that his chance of making a single shot is 20%. What is the expected value of Bob's prize for this promotion?

Solution
Using the binomial probability theorem, the probability of hitting exactly k shots out of n when the probability of hitting a particular shot is p (and the probability of hitting each shot is independent of the others) is $\binom{n}{k}p^k(1-p)^{n-k}$. Thus, we can just sum over k = 0, 1 to find the probability he doesn't get the $1000 and subtract this from 1 to get the probability that he does get the $1000:

$$P(Bob\ makes\ at\ least\ 2\ shots) = 1 - P(Bob\ makes\ 1\ shot) - P(Bob\ makes\ 0\ shots)$$

$$= 1 - \binom{9}{1}(.2)(.8)^8 - \binom{9}{0}(.8)^9 = 0.56379$$

From here, we calculate the expected prize as follows:

$$E(Bob's\ prize) = P(Bob\ makes\ at\ least\ 2\ shots) * Winnings = .56379 * \$1000$$
$$= \$\mathbf{563.79}$$

Set #2

1) Consider a variant of the Monty Hall Problem with n doors and 1 car behind a single door and goats behind the rest, where k (k < n-1) doors are revealed before you choose whether or not to switch. What is the probability of winning if you stay? What is the probability of winning if you switch?

Solution

Your probability of winning if you stay is $\frac{1}{n}$, the probability that you chose the correct door the first time around. If you switch, then you can lose in two ways: picking the winning door first, OR switching to a losing door instead of the winning door. This means that the only way we can win given that we switch is to pick a wrong door first, and then switch to the correct one. We multiply these probabilities, with the second conditional on the first event to get our result:

$$p(win|switch) = \frac{n-1}{n} \times \frac{1}{n-k-1}$$

Unfortunately, this does not reduce at all in the general case. For the original Monty Hall problem, we have n = 3 and k = 1, plugging in these two values into the formula above does indeed give the correct probability of winning $\frac{2}{3}$ of the time, given that we switch.

2) A factory manufactured 1,000,000 cell phones in 2012, of which 20,000 were defective. In the factory there are two assembly lines, A and B, responsible for manufacturing all of the phones. Assembly line A manufactured 200,000 phones total and assembly line B manufactured 12,000 defective cell phones. Determine the following probabilities:

a) Given that a cell phone was manufactured in assembly line A, what is the probability that it is defective? Give your answer as a percentage.

b) Given that a cell phone was manufactured in assembly line B, what is the probability that it is defective? Give your answer as a percentage.

Solution

a) We can infer that 20,000 – 12,000 = 8,000 defective phones were made in assembly line A. Thus, the desired probability is $\frac{8000}{200000}$= **4%**

b) We can infer that 1,000,000 – 200,000 = 800,000 phones were made in assembly line B. Thus, the desired probability is $\frac{12000}{800000}$ = **1.5%**

3) A bag of popping corn contains 2/7 white kernels and 5/7 yellow kernels. Only 2/5 of the white kernels will pop, whereas 5/9 of the yellow ones will pop. A kernel is selected at random from the bag, and pops when placed in the popper. What is the probability that the kernel selected was white?

Solution
Let W = kernel is white and K = kernel popped.

$$P(W|K) = \frac{P(K\,|W)P(W)}{P(K)}$$

$$P(K) = P(K|W)P(W) + P(K|\bar{W})P(\bar{W}) = \frac{2}{5} * \frac{2}{7} + \frac{5}{9} * \frac{5}{7} = \frac{23}{45}$$

$$\frac{P(K\,|W)P(W)}{P(K)} = \frac{\frac{2}{5} * \frac{2}{7}}{\frac{23}{45}} = \mathbf{\frac{36}{161}} = .224 = 22.4\%$$

4) Mikey is taking a matching quiz with 4 items on it. Unfortunately, Mikey didn't study and he completely guesses the answers, knowing that each of the four words will match to a different definition of the four given. What is the chance that Mikey gets all four definitions incorrect? What is the chance that he gets precisely 3 of the definitions correct? (We assume he matches each word to a unique definition listed.)

Solution
There are 24 possible answers Mikey can give corresponding to the 4! orderings of the definitions he can give. Let's count how many of them correspond to Mikey missing all 4 questions. Without loss of generality, assume that the correct answers are just 1, 2, 3 and 4. Here are the permutations of answers Mikey can give that don't match any of those answers, listed in lexicographical order:

2, 1, 4, 3
2, 3, 4, 1
2, 4, 1, 3
3, 1, 4, 2
3, 4, 1, 2
3, 4, 2, 1
4, 1, 2, 3
4, 3, 1, 2
4, 3, 2, 1

Thus, Mikey's chance of missing all 4 questions is $\frac{9}{24} = \mathbf{\frac{3}{8}}$.

It's impossible for Mikey to get 3 questions correct on the matching quiz. Thus the probability this occurring is **0.** If three of them were to match, by default the fourth answer would also. Thus, every permutation (possible answer given the rules) corresponds to either 0 correct, 1 correct, 2 correct or 4 correct.