COT 3100 Recitation: Counting 2 Problems Solutions

Set #1

1) How many 15 letter arrangements of 5 A's, 5 B's and 5 C's have no A's in the first 5 letters, no B's in the next 5 letters and no C's in the last 5 letters?

Solution

We want no A's in the first five letters, no B's in the next five letters and no C's in the last five letters.

Let's first try to fill the first five letters. We must only have B's and C's. We can choose the following:

0 B's, 5 C's
1 B, 4 C's
…
5 B's 0 Cs

Let's say we choose k B's for the first five slots. We can do this in $\binom{5}{k}$ ways. Then, the C's are forced into the other slots.

Now, the remaining 5-k B's must go in slots 11 through 15. Thus, we can choose the location for these B's in $\binom{5}{5-k} = \binom{5}{k}$ ways.

Finally, we must fill the middle 5 slots with A's and C's. Specifically, we have already placed 5 - k C's in the first 5 slots and placed k A's in the last five slots. So, really, we can just choose k out of 5 slots for the As in $\binom{5}{k}$ ways, and then the C's are forced.

We can break up our counting into 6 batches for each possible value of k, the number of B's in the first five slots. Thus, the answer is $\sum_{i=0}^{5} \binom{5}{k}^3 = 1^3 + 5^3 + 10^3 + 10^3 + 5^3 + 1^3 = \mathbf{2252}$.

2) Call a number "prime-looking" if it is composite but not divisible by 2, 3 or 5. The three smallest prime-looking numbers are 49, 77 and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are less than 1000?

Solution
Numbers can be in these categories:

1) Prime (size 168)
2) Divisible by 2, 3 or 5, **BUT NOT PRIME**
3) Composite Looking
4) 1 (size 1)

All 1000 positive integers from 1 to 1000 must fit in one of these 4 categories. What we know about each set's size is written above in yellow.

Strategy: Find out how many items in set #2 and then use subtraction to figure out the final answer.

Count all values divisible by 2, 3 or 5 via the inclusion-exclusion principle

$$\left\lfloor\frac{1000}{2}\right\rfloor + \left\lfloor\frac{1000}{3}\right\rfloor + \left\lfloor\frac{1000}{5}\right\rfloor - \left\lfloor\frac{1000}{6}\right\rfloor - \left\lfloor\frac{1000}{10}\right\rfloor - \left\lfloor\frac{1000}{15}\right\rfloor + \left\lfloor\frac{1000}{30}\right\rfloor =$$

$$500 + 333 + 200 - 166 - 100 - 66 + 33 = 734$$

The number of values in group 2 is 734 - 3 = 731, because we have to subtract out 2, 3 and 5, which are not part of group 2.

Final answer = 1000 - 168 - 731 - 1 = **<u>100</u>**

3) A faulty car odometer proceeds from digit 3 to digit 5, always skipping the digit 4, regardless of position. For example, after traveling one mile the odometer changed from 000039 to 00050. If the odometer now reads 002005, how many miles has the car actually traveled?

Solution
Basically, our odometer has 9 possible digits for each slot, not 10. So, it's like a number in base 9… 2005 really means that the 4^{th} slot from the right moved twice, and it moves twice with the other slots to the right of it go through 9^3 positions. So after the car has traveled 2 x 9^3 = 1458 miles, the odometer reads 002000. Then the odometer counts 002001, 002002, 002003 and 002005. It goes ***four*** more miles to get to 002005, so the total miles traveled is 1458 + 4 = **1462.**

4) Call a set of integers *spacy* if it contains no more than one out of any three consecutive integers. How many subsets of {1, 2, 3, …, 12}, including the empty set, are spacy?

Solution
Combinations with repetition tells us that the equation

$x_1 + x_2 + \ldots + x_r = n$ has exactly $\binom{n+r-1}{r-1}$ non-negative integer solutions for $(x_1, x_2, \ldots, x_r)$.

First observation: we can't have more than 4 items from the set, since after you take an item you have to skip 2 more after it.

Sets of size 0, 1, 2, 3, and 4

Size 0: 1 set (empty set)
Size 1: 12 sets (any of the 12 sets of 1 item are spacey)

Size 2: We pick our 2 items but then have to separate them by at least 2:

(not taken bin #1) X (not taken bin with 2 already, bin #2) X (not taken bin bin #3)

The x's are 2 items and we also have to have 2 items in between the x's. This means that we have 8 more items free to place in the the 3 non-taken bins.

Let x be the size of bin 1, y be the size of bin 2 and z be the size of bin 3. We know that

$x + y + z = 8$, and x, y and z are non-negative integers.

There are $\binom{8+3-1}{3-1} = \binom{10}{2} = 45$ ways to place these 8 items. So there are precisely 45 spacey sets with 2 items in them.

Consider the following solution to the equation:

$x = 4$, $y = 1$ and $z = 3$

This means that {1,2,3,4} are NOT in the set, since x = 4
5 is in the set
This means that {6,7,8} are NOT in the set.
9 is in the set
This means that {10,11,12} are NOT in the set, since z = 3

Size 3: We pick our 3 items but then have to separate them by at least 2:

(bin #1) X (bin #2) X (bin #3) X (bin #4)

Two things must be placed in both bin #2 and bin #3. So we have already accounted for 3 X's and 4 spacing items, leaving 5 other times to be placed amongst 4 bins, which can be done in $\binom{5+4-1}{4-1} = \binom{8}{3} = 56$ spacy subsets of size 3.

Size 4: We pick our 4 items but then have to separate them by at least 2:

(bin #1) X (bin #2) X (bin #3) X (bin #4) X (bin #5)

Two things must be placed in both bin #2,bin #3 and bin #4 So we have already accounted for 4 X's and 6 spacing items, leaving 2 other times to be placed amongst 5 bins, which can be done in $\binom{2+5-1}{5-1} = \binom{6}{4} = 15$ spacy subsets of size 4.

Final answer = 1 + 12 + 45 + 56 + 15 = 13 + 101 + 15 = **<u>129</u>**.

<u>Alternate Solution Idea</u>

Let c_k represent the number of spacy subsets of the set {1, 2,…, k}.
$c_1 = 2$, $c_2 = 3$, $c_3 = 4$ (initial conditions)

Now, let's see if we can come up with a recurrence relation for c_k.

{1, 2, 3, 4, 5…, k} → spacy subsets WITH k
spacy subsets WTHOUT k

If k is NOT in the subset, then we are just looking any spacey subset from the set {1,2,…,k-1}. There are exactly c_{k-1} of these!!!

Alternatively, if we take item k, we are forced not to take k-1 and k-2. So we are forced to pick the remaining items as any spacey subset from the set {1,2,3,…,k-3}. By definition, we can do this in c_{k-3} ways.

Thus, $c_k = c_{k-1} + c_{k-3}$.

Now, just use this formula to build up the answers upto c_{12}:

$c_1 = 2$, $c_2 = 3$, $c_3 = 4$, $c_4 = 6$, $c_5 = 9$, $c_6 = 13$, $c_7 = 19$, $c_8 = 28$, $c_9 = 41$, $c_{10} = 60$, $c_{11} = 88$, $c_{12} =$ **<u>129.</u>**

Set #2

1) A line passes through A(1, 1) and B(100,1000). How many other points with integer coordinates are on the line and strictly between A and B?

Solution

The slope will indicate the relationship between x and y on the line segment AB, so let's calculate that: $\frac{1000-1}{100-1} = \frac{999}{99} = \frac{111}{11}$. This is the slope in lowest terms.

Since A is a lattice point, and the slope of the line in lowest terms is $\frac{111}{11}$, the "next" lattice point on the line segment is (1 + 11, 1 + 111) = (12, 112).

The slow way of solving this problem would be to simply list every point in this manner, adding 11 to the x-coordinate and 111 to the y-coordinate of the previous point calculated. If we do it this way we get the list: (12, 112), (23, 223), (34, 334), (45, 445), (56, 556), (67, 667), (78, 778), and (89, 889), so there are <u>**8**</u> such points.

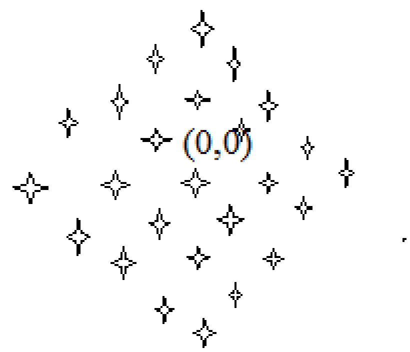
In fact, we can make a more general finding. Given any arbitrary lattice point A(x_A, y_A) and lattice point B(x_B, y_B), the number of lattice points strictly inside the line segment AB (not including A or B), will always equal gcd(x_B - x_A, y_B - y_A) - 1. (Note: of course, points A and B must be distinct.) The reason is, that when we divide by the gcd, this shows us how many times we "hop" between successive lattice points. Of these hops, only the last one, where we land onto point B, doesn't count.

2) An object in the plane moves from one lattice point to another. At each step, the object may move one unit to the right, one unit to the left, one unit up, or one unit down. If the object starts at the origin and takes a ten-step path, how many different points could be the final point?

Solution

Tracing out the points we can reach, we get to any lattice point whose sum of coordinates is even and $|x| + |y| \leq 10$. When you draw this out, you get a square, rotated by 45 degrees with 11 lattice points on each side, so the final answer is 11^2 = <u>**121.**</u>

Another way of viewing the picture is to look at it scanning vertically, at each individual x value. At x = -10, there is one point. At x = -9, there are 2 points. At x = -8 there are 3 points, etc. This gives us the sum: $(\sum_{i=1}^{11} i) + (\sum_{i=1}^{10} i) = \frac{11\times12}{2} + \frac{10\times11}{2} = \frac{11}{2}(10+12) = \frac{11\times11\times2}{2} = 11^2$ = **121.**



Move 4 times.

This is a square with side length 5

More generally, if I move 2n times, I get a square (tilted 45 deg), of side length (2n+1). So we reach any of (2n+1)(2n+1) lattice points.

3) How many ordered quadruplets (a_1, a_2, a_3, a_4) of non-negative integers, where at least one of the integers is even, satisfy the equation, $a_1 + a_2 + a_3 + a_4 = 100$?

Solution

In general, as we learned in class, the number of non-negative integer solutions to the equation

$$x_1 + x_2 + x_3 + \ldots + x_r = n,$$

is $\binom{n+r-1}{r-1}$.

We could apply this formula to our problem, but then we would be counting some solutions were each of the given variables is odd. Why not try to just subtract these out???

Total number of solutions is combinations with repetition with n = 100 and r = 4, so there are a total of $\binom{100+4-1}{4-1} = \binom{103}{3}$.

Now, we must count how many of these solutions are an ordered quadruplet of 4 odd integers.

Let a_1, a_2, a_3 and a_4 be odd positive integers. Therefore, there exists non-negative integers x, y, z and w such that $a_1 = 2x + 1$, $a_2 = 2y + 1$, $a_3 = 2z + 1$ and $a_4 = 2w + 1$. How substitute this into the given equation:

$$a_1 + a_2 + a_3 + a_4 = 100$$

$$(2x+1) + (2y+1) + (2z+1) + (2w+1) = 100$$

$$2(x + y + z + w) = 96$$

$$x + y + z + w = 48$$

Since x, y, z and w are arbitrary non-negative integers, we can just apply the combinations with repetition formula with n = 48 and r = 4, so that there are a total of $\binom{48+4-1}{4-1} = \binom{51}{3}$ solutions where all four variables are odd.

It follows that our final answer, the number of non-negative integer solutions to the given equation where at least one of the four variables is even is $\boldsymbol{\binom{103}{3} - \binom{51}{3}}$**.**

4) How many non-negative integer solutions are there to the equation

$$a + b + c + d + e + f + g + h + i + j \leq 50?$$

Solution

As taught in class, for any solution to this equation, we can calculate the difference from 50 of the sum of all the variables. In fact, for each solution to this equation, there is a one to one correspondence to solutions to the equation

$$a + b + c + d + e + f + g + h + i + j + k = 50$$

where k is also a non-negative integer. Any solution to the inequality maps to a unique solution to the equality above, and any solution to the equality maps to a unique solution to the inequality. Thus, we can answer the question by answering how many non-negative solutions there are to the second equation, which is simply, $\binom{50 + 11 - 1}{11 - 1} = \boldsymbol{\binom{60}{10}}$.