COT 3100 Recitation: Counting 1 Problems

<u>Set #1</u>

1) How many three digit numbers are not divisible by 5, have digits that sum to less than 20, and have the first digit equal to the third digit?

<u>Solution:</u>
Let's restrict our search by looking at 3 digit numbers that start and end with the same digit. This digit must be anything 1 through 9, except 5, since any number ending in 5 is divisible by 5. There are 8 x 10 = 80 of these, since there are 8 choices for the units digit and 10 choices for the tens digit. (The hundred's digit is chosen, once the units digit is chosen.) But, of these 80, we must subtract out the ones that have a sum of digits 20 or greater. In order for this to occur, the units digit must at least be 6. Here is a chart of what gets subtracted out:

Units Digit	Max Tens Digit	Numbers to Subtract out
6	7	2 (8 and 9)
7	5	4 (6,7,8 and 9)
8	3	6 (4,5,6,7,8 and 9)
9	1	8 (2,3,4,5,6,7,8 and 9)

Thus, our final answer is 80 - 2 - 4 - 6 - 8 = **60.**

2) Find the sum of the digits in all of the integers from 1 to 10000, inclusive.

<u>Solution</u>
Consider summing all of the digits in the numbers from 0000 to 9999. It should be fairly clear that each digit appears in each slot exactly 1000 times over the course of the 10000 numbers, since each of the 10 digits cycles through each slot (ones, tens, hundreds, thousands) the same number of times, in complete cycles. (The ones digit has 1000 cycles of 0,1,2,3,4,5,6,7,8,9, the tens digit is 100 cycles of 10 0s, 10 1s, 10 2s, …, 10 9s. etc.) Thus, for each digit slot, we want to count each digit 0 through 9 appearing 1000 times. There are 4 slots, times 1000 repetitions, times 45, the sum of the digits from 0 to 9. This gives us a sum of 4 x 1000 x 45 = 180,000. Add to this the 1 from the most significant digit of 10,000 and we get a final total of **<u>180,001.</u>**

3) One thousand unit cubes are put together into a 10 x 10 x 10 cube and the surface of this 10 x 10 x 10 cube is painted. Then, the unit cubes are pulled apart. How many of the 1000 unit cubes have at least one face painted?

Solution

We can calculate this by counting the number of unpainted unit cubes. The unpainted unit cubes form an 8 x 8 x 8 cube within the painted ones, so the number of painted cubes is $10^3 - 8^3 =$ **488.**

4) How many different integers in between 100 and 999 have either three increasing digits or three decreasing digits? (1990 ahsme #17))

Solution

To get three increasing digits, choose 3 digits out of the set {1,2,3,4,5,6,7,8,9}. We can do this in $\binom{9}{3} = 84$ ways.

To get three decreasing digits, choose 3 digits out of the set {0,1,2,3,4,5,6,7,8,9}. We can do this in $\binom{10}{3} = 120$ ways.

It follows that the final answer is 84 + 120 = **204.**

Set #2

1) How many ordered pairs of positive integers (x, y) satisfy the equation 3x + 5y = 501?

Solution

We quickly get one solution by plugging in y = 0 since 501 is divisible by 3, which is (167, 0). Of course, we aren't actually supposed to count this solution since x and y aren't positive. We know that all integer solutions can be expressed as:

{(x, y) | x = 167 - 5c, y = 3c, for all integers c}

Thus, we need to know for how many integers c, 167 - 5c > 0 AND 3c > 0. The former inequality says that c < 167/5, it follows that since c is an integer c ≤ 33, since 33 is the largest integer less than 167/5. Thus, we are looking for the number of integers c that satisfy 0 < c ≤ 33. There are **33** such values of c, thus, there are 33 solutions to the given equation that satisfy the constraints given. These solutions range from (162, 3) to (2, 99).

2) Belinda writes down the positive integers in order starting at 1. What is the 2020^{th} digit that she will write down? (For example, she writes 1,2,3,4,5,6,7,8,9,10,11,12,13 to begin, so the 15^{th} digits she writes down is 2, the second digit when writing down 12.)

Solution

There are 9 one digit numbers, totaling 9 digits.
There are 90 two digit numbers totaling 180 digits.
Thus, after writing 189 digits, we have left 2020 - 189 = 1831 digits left to write.
Each of the following 900 numbers will have 3 digits each.
Since $\left\lfloor \frac{1831}{3} \right\rfloor = 610$, then we will completely write 610 three digit numbers. These numbers are 100 through 709. Thus, After writing 709, Belinda has written 189 + 610*3 = 2019 digits. This means that the 2020^{th} digit that Belinda writes is **7**, which she is writing the number 710.

3) Mr. and Mrs. Zeta want to name their child so that the child's three initials (first, middle, last) are in alphabetical order with no repeated letters. How many different sets of initials could the child have?

Solution

The last initial is Z. We can choose the other two initials from 25 choices, A through Y. Once we choose 2 distinct letters, the earlier one alphabetically is forced to be the first initial and the other one is forced to be the second initial. So, the answer is just the number of ways to choose 2 letters out of 25, which is $\binom{25}{2} = \mathbf{300}$.

4) How many integers with 4 different digits in between 1000 and 9999 are there such that the absolute value of the difference between the first and last digit is 2?

Solution

There are eight different ways to get 2 with the absolute value of two different digits:

$$|2-0|=2,\ |3-1|=2,\ |4-2|=2,\ |5-3|=2$$
$$|6-4|=2,\ |7-5|=2,\ |8-6|=2,\ |9-7|=2$$

Because of the absolute value, these numbers can be interchanged giving us 16 possible ways. However, the range we are working with is between 1000 and 9999. This means that the first digit can only be 9 digits (1-9) and the last digits can be 10 digits (0-9). 2 – 0 is possible but 0 – 2 is not within our range so there are 15 possible ways.

For the second and third digits, they must be different from each other and the first and last digits. Since at most, the last digits have 10 possible choices and there is only one possible choice for the first digit, there are only 8 possible choice for the second digit and 7 possible choice for the third digit.

Thus, 15 * 8 * 7 = **840** integers