COT 3100 Recitation : Use of Factorization of Integers, Algebraic Expressions

Solutions

Set #1

1) The zeros of the function f(x) = x^2 - ax + 2a are integers. What is the sum of the possible values of a?

Solution

The two roots of the equation are: $\frac{a \pm \sqrt{a^2-8a}}{2}$. If both of these are two be integers, then the difference between the two roots must be integral, which means that $\sqrt{a^2-8a}$ must be an integer, which means that $a^2 - 8a$ must be a perfect square. Without loss of generality, there exists a non-negative integer b such that:

$$a^2 - 8a = b^2$$
$$a^2 - 8a + 16 - 16 = b^2$$
$$(a-4)^2 - 16 = b^2$$
$$(a-4)^2 - b^2 = 16$$
$$(a-4+b)(a-4-b) = 16$$

Since b ≥ 0, we have six possibilities:

$$a-4+b = 4, a-4-b = 4$$
$$a-4+b = -4, a-4-b = -4$$
$$a-4+b = 8, a-4-b = 2$$
$$a-4+b = -2, a-4-b = -8$$
$$a-4+b = 16, a-4-b = 1$$
$$a-4+b = -1, a-4-b = -16$$

The first leads to the solution b = 0, a = 8. The second leads to the solution b=0, a=0. The third leads to the solution b = 3, a = 9. The fourth leads to the solution b = 3, a = -1. The last two lead to a non-integer solutions of a and b. Plugging back each of these values of 8 yields the following quadratics: x^2 - 8a + 16 = 0, x^2 = 0, x^2 - 9x + 18 = 0, and x^2 + x - 2 = 0.These have the roots (4, 4), (0, 0), (3, 6), and (1, -2), respectively, so all four are valid.

It follows that the sum of the possible values of **a is 8 + 0 + 9 + -1 = 16.**

2) What is the smallest possible positive integer x such that 1260x = N^3, where N is some positive integer?

Solution

1260x = N^3

$2^2 3^2 5^1 7^1 x = N^3$

N must have 2, 3, 5 and 7 in its prime factorization. At a minimum N = 2 x 3 x 5 x 7 = 210. Substituting, we have:

$2^2 3^2 5^1 7^1 x = 2^3 3^3 5^3 7^3$

x = $2^1 3^1 5^2 7^2$ = **7350**

3) The number 10!, when written in base 12, ends in how many zeroes? (Challenge: figure out a fast algorithm to solve this problem when the factorial can be quite large and the base is anything that is easy to prime factorize.)

Solution

The brute force method:

$10! = 3268800_{10}$

Base conversion (until we hit a non-zero remainder):

$3268800/12 = 302400\ R0$

$302400/12 = 25200\ R0$

$25200/12 = 2100\ R0$

$2100/12 = 175\ R0$

$175/12 = 14\ R7$, we can stop here.

When converted to base 12, **10! ends in four zeros.**

In general, we see that for a given prime p, p divides into n! $\sum_{i=1}^{n} \left\lfloor \frac{n}{p^i} \right\rfloor$ times. To see this, note that if we divide p into n and take the floor, this tells us the number of times we "cancel" p for each multiple of p in the written product of n!. But, in doing so, we DIDN'T cancel 2 copies of p from the values p^2, $2p^2$, etc. So, we want to divide by p^2 to get the second copies of the prime p from those terms. We continue in this fashion. We can stop the sum well before n - namely, we can stop as soon as i is such that $p^i > n$.

Now, to apply it to this problem, 12 = $2^2 3$. Thus, we can calculate the number of times 2 divides into 10! using the formula above and do the same for 3. Then we see which of the two items is the limiting factor. For example, 2 divides into 10! 5 + 2 + 1 = 8 times, so 2^2 divides into 12! 4 times. 3 divides into 10! 3 + 1 = 4 times. Thus, both primes limit the number of times 12 divides evenly into 10! and thus there are 4 factors of 12 in 10!, meaning that in base 12, 10! ends in **4 zeroes.**

4) Let A, M and C be non-negative integers such that A + M + C = 12. What is the maximum value of AMC + AM + MC + AC?

Solution

This is a creative step. When you see the terms listed, you see all combinations of size 2 and size 3 of the given terms. One way to list out all combinations added together is multiplying each term plus one:

(A + 1)(M + 1)(C + 1) = AMC + AM + MC + AC + A + M + C + 1
= (AMC + AM + MC + AC) + 12 + 1

Thus, we have

(AMC + AM + MC + AC) = (A + 1)(M + 1)(C + 1) – 13

Our goal then, is to maximize the product (A + 1)(M + 1)(C + 1). More clearly, let A' = A + 1, M' = M + 1 and C' = C + 1. Then our goal is to maximize A'M'C' with the constraint that A' + M' + C' = 15. The generalization of the arithmetic-geometric mean states that the geometric mean of a set of positive numbers is always less than or equal to their arithmetic mean. In this case, the arithmetic mean of A', B' and C' is 5. Thus, $\sqrt[3]{A'M'C'} \leq 5$. Alternatively, we have $A'M'C' \leq 5^3 = 125$. Equality is achieved (and this can be seen pretty easily) when each term is equal. Thus, by setting A' = 5, M' = 5 and C' = 5, we achieve a maximum of 125 for A'M'C'. It follows that the maximum value for the quantity in question is 125 – 13 = **112.**

Set #2

1) Both roots of the quadratic equation $x^2 - 63x + k = 0$ are prime numbers. How many possible values for k are there?

Solution

In general, the sum of the roots of a quadratic equation with a leading coefficient of 1 is the negative of the x coefficient. To see this, note that any quadratic of this form can also be written as $(x - r)(x - s)$, where *r* and *s* are the two roots of the equation. Multiplying this out we get:

$$x^2 - rx - sx + rs = x^2 - (r + s)x + rs$$

From here, if we equate this to the regular quadratic form of the equation, we see that the sum of the roots (r+s) is the opposite of the *x* coefficient.

Thus, the sum of the two roots of the equation is 63. One possible solution is that the two prime numbers in question are 2 and 61, respectively. There are no other possibilities for the roots because 2 is the only even prime number and if the two roots sum to be odd, precisely one must be even. **Thus, there is only one possible value of k, 122.**

2) How many positive integers less than 200 have an odd number of positive integer divisors?

Solution:

Factors come in pairs in all cases except one, perfect squares. The question is thus asking us how many perfect squares there are between 0 and 200. A listing of the perfect squares is as follows: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196. There are **14** perfect squares less than 200.

3) The number 2^{48} - 1 is exactly divisible by two numbers between 60 and 70. What are they?

Solution

Let's factor the expression:

2^{48} - 1 = $(2^{24} - 1)(2^{24} + 1)$
= $(2^{12} - 1)(2^{12} + 1)(2^{24} + 1)$
= $(2^{6} - 1)(2^{6}+1)(2^{12} + 1)(2^{24} + 1)$

Two of the factors listed above are **63 and 65.** Since it's given that there are only 2 divisors between 60 and 70, these must be the two.

4) Integers x and y with $x > y > 0$ satisfy $x + y + xy = 80$. What is x?

Solution

Add 1 to both sides of the equation to yield:

$xy + x + y + 1 = 81$

$(x+1)(y+1) = 81$

If two integers multiply to 81 and they are distinct, those two integers must be either 1 and 81 or 3 and 27. But, if we set y+1 to 1, y would be 0 and this isn't allowed. It follows that x+1 = 27 and y+1 = 3. Thus, **x = 26.** (Also, we can deduce that y = 2.)