COT 3100 Recitation: Arithmetic Geometric Series Solutions

Set #1

1) An arithmetic series has a fourth term equal to 13 and its tenth term equal to 31. What is the sum of the first 100 terms of the series?

Solution

First, get the common difference by using the two known terms:

$a_{10} – a_4 = (10 – 4)d = 31 – 13 \rightarrow 6d = 18 \rightarrow d = 3$

Then, obtain a_1 from the value of a_4:

$a_4 = a_1 + (4-1)d$
$13 = a_1 + 3(3)$
$a_1 = 4$

Also, obtain a_{100}:

$a_{100} = a_1 + 99d = 4 + 99(3) = 301$

$$S_{100} = \frac{(a_1+a_{100})}{2} \times 100 = \frac{(4+301)}{2} \times 100 = \frac{30500}{2} = \mathbf{15250}$$

2) In a given arithmetic sequence the first term is 2, the last term is 29 and the sum of the terms is 155. What is the common difference of the sequence?

Solution

$S_n = \frac{(a_1+a_n)}{2} \times n = \frac{(2+29)}{2} \times n = 155$. It follows that $n = \frac{310}{31} = 10$. Now, solve for d:

$a_{10} - a_1 = 9d$
$29 - 2 = 9d$
$9d = 27$
$d = \mathbf{3}$

3) Find the value of $a_2 + a_4 + a_6 + \ldots + a_{98}$, if $a_1, a_2, a_3, \ldots$, is an arithmetic progression with common difference 1 and the sum of the first 98 terms of the sequence is 137.

Solution

Let $X = a_2 + a_4 + \ldots + a_{98}$ and Let $Y = a_1 + a_3 + \ldots + a_{97}$. Note that $a_1 = a_2 - 1$, $a_3 = a_4 - 1$, etc. It follows that:

$$Y = (a_2 - 1) + (a_4 - 1) + \cdots + (a_{98} - 1) = (a_2 + a_4 + \cdots + a_{98}) - 49 = X - 49$$

We also know that $X + Y = 137$

Substitute for Y in this equation above:

$X + X - 49 = 137$
$2X = 186$
$X = \mathbf{93}$

4) The geometric series $a + ar + ar^2 + \ldots$ has a sum of 7, and the terms involving odd powers of r have a sum of 3. What is a + r?

Solution

Let $S = a \quad + ar^2 + ar^4 + \ldots$
Let $T = ar + ar^3 + ar^5 + \ldots = r(a + ar^2 + ar^4 + \ldots) = rS = 3$.

Recall that S + T = 7, and T = 3, so it follows that S = 4. Then, we can solve for r:

$4r = 3 \rightarrow r = 3/4$.

We solve for a as follows: $S = 7 = \frac{a}{1-3/4} \rightarrow 4a = 7 \rightarrow a = \frac{7}{4}$.

The desired result is $\frac{7}{4} + \frac{3}{4} = \mathbf{\frac{5}{2}}$.

Set #2

1) A sequence of three real numbers forms an arithmetic progression with the first term 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the smallest possible value for the third term of the geometric progression.

Solution

Let the terms be a, b and c, with a = 9, b = 9 + d and c = 9 + 2d, where d is the common differences of the arithmetic sequence. Let a' = a = 9, b' = b + 2 = 11 + d and c' = 20 + c = 29 + 2d. Setting equations to solve for the common ratio of the geometric sequence we see that:

$$\frac{d+11}{9} = \frac{2d+29}{d+11}$$

Cross multiply (valid as long as d isn't -11, which would be impossible if the second sequence is a valid geometric sequence with a non-zero common ratio) to yield:

$$18d + 261 = d^2 + 22d + 121$$
$$d^2 + 4d - 140 = 0$$
$$(d+14)(d-10) = 0$$

It follows that the common difference is either -14 or 10. Thus, the original arithmetic sequence is either 9, -5, -19 or 9, 19, 29. The corresponding geometric sequences would be 9, -3, 1 and 9, 21, 49. It follows that the smallest possible value for the third term of the geometric sequence is **1.**

2) The sequence $a_1, a_2, \ldots, a_n$, $n > 1$ is an arithmetic sequence with $a_1 = 10$, $a_n = 50$, and a common difference, d, that is a positive integer. What is the sum of all the possible values of d?

Solution

Observations

1) d(n) = (50 – 10)/(n-1)
2) The minimum value of n is 2
3) The maximum value of d, achieved at n = 2, is d(n) = (50 – 10)/(2-1) = 40
4) The minimum value of d is 1
5) The maximum value of n, achieved at d = 1, is 1 = (50 – 10)/(n-1) = 41
6) For d(n) to be a positive integer, (n-1) must be a factor of 40.

Therefore:

sum = d(2) + d(3) + d(5) + d(6) + d(9) + d(11) + d(21) + d(41)
sum = 40/1 + 40/2 + 40/4 + 40/5 + 40/8 + 40/10 + 40/20 + 40/40
sum = 40 + 20 + 10 + 8 + 5 + 4 + 2 + 1 = **90.**

3) In an arithmetic sequence the sum of the 72^{nd} term and the 112^{th} terms is 22. What is the sum of the first 183 terms of this sequence? To prove that the sequence is not unique, provide the first term and common difference of two difference sequences that satisfy the requirements given in this problem. Make sure both of your sample sequences have non-zero common differences.

Solution

Let the nth term of the sequence be a_n, the common difference of the sequence be d and express a_{72} and a_{112} in terms of a_{92}:

$a_{72} = a_{92} + (72 – 92)d = a_{92} - 20d$
$a_{112} = a_{92} + (112 – 92)d = a_{92} + 20d$

Adding these two terms we get: $a_{72} + a_{112} = a_{92} - 20d + a_{92} + 20d = 2a_{92} = 22$.

Thus, $a_{92} = 11$.

Express every term from a_1 through a_{183} in terms of a_{92} and you get:

$$\sum_{i=1}^{183} a_{92} + (i - 92)d$$

Notice that the term (i-92) ranges from -91 to positive 91, so all terms cancel with one another. Formally, we have:

$$\sum_{i=1}^{183} a_{92} + (i - 92)d = \sum_{i=1}^{183} a_{92} + \sum_{i=1}^{183}(i - 92)d$$
$$= 11(183) + d\sum_{i=1}^{183}(i - 92)$$

Here we will split the sum in half, omitting the term for i = 92, since it’s 0.

$$= 11(183) + d\left(\sum_{i=1}^{91}(i - 92) + \sum_{i=93}^{183}(i - 92)\right)$$

Here we rewrite the first sum by pulling out a negative sign and reindex the second sum:

$$= 11(183) + d\left(-\sum_{i=1}^{91}(92 - i) + \sum_{i=1}^{91}(i)\right)$$

We notice that we can reexpress the first sum equivalently by getting rid of the 92:

$$= 11(183) + d\,(-\sum_{i=1}^{91}(i) + \sum_{i=1}^{91}(i))$$

Finally, the two sums cancel and we get:

$$= 2013 + d(0) = \mathbf{2013}$$

4) a, b and c form an arithmetic sequence with a non-zero common difference. If a is increased by 1, the resulting sequence is a geometric sequence. Alternatively, if c is increased by 2, the resulting sequence is also a geometric progression. What is b?

<u>Solution</u>
Let the common difference of the arithmetic sequence be d. It's three terms are b - d, d and b + d. The two sequences that are geometric sequences are:

b - d + 1, b, b + d AND
b - d, b, b + d + 2

This gives us the following ratios:

$\frac{b}{b-d+1} = \frac{b+d}{b}$ and $\frac{b}{b-d} = \frac{b+d+2}{b}$

Cross multiplying in both situations yields:

$b^2 = (b+d)(b-d) + b + d$ and $b^2 = (b+d)(b-d) + 2b - 2d$

If we subtract the two equations, we get $0 = b - 3d$, so we get that $b = 3d$.

Now, we know that the arithmetic progression was 2d, 3d, 4d. The two geometric progressions are 2d+1, 3d, 4d and 2d, 3d, 4d+2. Plug back into the ratios to solve for d:

$\frac{3d}{2d+1} = \frac{4d}{3d} \rightarrow \frac{3d}{2d+1} = \frac{4}{3} \rightarrow 9d = 8d + 4 \rightarrow d = 4.$

It follows that **<u>b = 12.</u>**