# Arithmetic/Geometric Series

*Arithmetic Series*

An arithmetic series is a series of numbers, where the difference between successive terms is the same. Here are a few examples of arithmetic series:

3, 5, 7, 9, 11, 13
1, 8, 15, 22
-4, -7, -10, -13, -16

We can completely define an arithmetic series by the following information:

1) $a_1$ – the first term (in general we say $a_k$ is the $k^{th}$ term)
2) d – the common difference
3) n – the number of terms.

For the first sequence listed above, $a_1 = 3$, $d = 2$ and $n = 6$.

Technically, the series can go forever, of course.

Here are things we'd like to be able to do given partial information about an arithmetic series:

1) Given a particular term $a_i$ and the common difference d, find any other term, $a_j$.

2) Given $a_1$, d and some value x, determine which term in the sequence equals x.

3) Find the sum of the first n terms of the series.

To solve the first task, observe that $a_j - a_i = (j - i)d$. Basically, to get to term j from term i, you have to "hop by d" exactly j – i times. Once you understand this, then the formula to solve for $a_n$ is fairly intuitive:

$a_n = a_1 + (n-1)d$

Also, item (2) is similar to item (1), just solving backwards. We would have:

$x = a_1 + (n-1)d \rightarrow n = \frac{x-a_1}{d} + 1$

To find the sum of an arithmetic sequence note that:

$$S = a_1 + a_2 + \cdots + a_{n-1} + a_n$$
$$S = a_n + a_{n-1} + \cdots + a_2 + a_1$$

If we look to add each column, we see that the first column adds up to $a_1 + a_n$. The interesting thing is that the second column does also:

$a_2 + a_{n-1} = a_1 + d + a_n - d = a_1 + a_n$

Intuitively, when we move from left to right one column, on the top we are adding d and on the bottom we are subtracting d, so these cancel out, leaving the sum of the two terms unchanged.

It follows then, that each of the n columns adds to $a_1 + a_n$, and we have:

$2S = n(a_1 + a_n)$, and
$S = \frac{n}{2}(a_1 + a_n)$

Typical problems given can usually be solved with some combination of these skills. Often times, you just have to set up equations with the given information and then the problem boils down to solving those equations (usually 2 equations with 2 unknowns).

<u>*Geometric Series*</u>
Geometric series are similar to arithmetic ones, except that the ratio between successive terms is constant. Here are examples of a some geometric series:

3, 6, 12, 24
2, 1, ½, ¼, 1/8, 1/16
3, 3/2, ¾, 3/8, …

We can completely define a geometric series by the following information:

1) $a_1$ – the first term (in general we say $a_k$ is the $k^{th}$ term)
2) r – the common ratio
3) n – the number of terms.

For the first series listed above, we have $a_1 = 3$, r = 2 and n = 4.

You'll notice that the third sequence has a … after it. This indicates that it goes forever. When we are adding numbers, when the ratio between successive terms has an absolute value less than 1, it turns out that the sum of an infinite number of terms converges to a value. This is never true of arithmetic series (except for all 0s). Thus, an additional problem we can solve for some geometric series is the sum of an infinite number of terms.

Here are the basic skills we'll need to solve problems with geometric series:

1) Given a particular term $a_i$ and the common ratio r, find any other term, $a_j$.

2) Given $a_1$, r and some value x, determine which term in the sequence equals x.

3) Find the sum of the first n terms of the series.

4) Find the sum of an infinite geometric series where $|r| < 1$.

When we hop from term to term in a geometric sequence, we multiply (instead of adding). It follows that $\frac{a_j}{a_i} = r^{j-i}$. Each hop is simply multiplying by r, and we want to hop j – i times.

Using this information, we can ascertain the typical formula shown to students:

$a_n = a_1 r^{n-1}$

For the second item, we can use this information to set up the following equation:

$\frac{x}{a_1} = r^{n-1}$ (solve for n)

Now, to find the sum, we have to do some work:

$$S = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1}$$

Multiply this equation through by r to get:

$$rS = a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1} + a_1 r^n$$

Line these equations up and subtract the second from the first:

$$\begin{aligned} S &= a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} \\ rS &= \phantom{a_1 +{}} a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} + a_1 r^n \end{aligned}$$

Notice that all but the first and last terms cancel, yielding:

$$S - rS = a_1 - a_1 r^n$$

$$S(1-r) = a_1(1-r^n)$$

$$S = a_1 \times \frac{(1-r^n)}{(1-r)} = a_1 \times \frac{(r^n - 1)}{(r-1)}$$

Notice that this work is only valid if $r \neq 1$. (But, a sequence with $r = 1$ is just a repetition of the same number, which can be added via regular multiplication.)

Finally, imagine situations where $|r| < 1$ and the series goes infinitely. The sum of the infinite number of terms would look similar to what we have above, except that the term $r^n$ will approach 0 and n goes to infinity. Thus, to get the sum of an infinite geometric series, just remove this term:

$$S_\infty = \frac{a_1}{1-r}$$

This is a really neat, clean formula!