COT 3100 Fall 2022 Homework #9 Solutions
Please Consult WebCourses for the due date/time

Note: Please *justify* your answers and why you use each formula. Please leave answers in factorials, combinations and powers. When possible, reduce fractions to lowest terms.

1) (5 pts) Johnny has rolled a pair of standard fair six-sided dice. Given that one of the two dice shows 3 dots, what is the probability that the sum of the number of dots showing on both dice is even?

Solution

Out of the 36 possible ordered pairs of rolls, 11 of them contain a 3. (These are rolls of the form (x, 3) or (3, x) where x is in the set {1,2,3,4,5,6}. Notice that we only count (3, 3) once, which is why there are 11 such rolls and not 12.)

Of these 11, the following sum to an even total: (3, 1), (3, 3), (3, 5), (1, 3), and (5, 3).

It follows that the desired probability is $\mathbf{\frac{5}{11}}$.

2) (8 pts) In class, we played a game where 3 distinct digits were hidden behind 3 "doors." The contestant would choose one of the doors and the number behind it was revealed. Based on this number, the contestant could either choose to stay with that door, or switch to another door, with the goal of selecting the door that had the lowest number behind it. We also showed in class that the optimal strategy for the game was to keep the same door if the number revealed was 3 or lower, and to switch otherwise. Change the game so that the numbers behind the 3 doors are distinct positive integers in between 1 and 30, inclusive. What is the largest integer k, such that if a contestant reveals k, their best chance of winning is staying with that door. For that value of k, what is the contestant's chance of winning, given that they stay with the same door? For that value of k, what is the contestant's chance of winning if they switch to a different door?

Solution

The probability of winning, if we pick k on our first turn, and stay with our choice is $\frac{\binom{30-k}{2}}{\binom{29}{2}}$.

The sample space is $\binom{29}{2}$, because once k is chosen, the other 2 numbers can be taken from the other 29 possible numbers (any of the 30 numbers except k).

The number of successes, when k is the smallest, is when both of the other two numbers are in the set {k+1, k+2, …, 30}. There are 30 – k numbers in this set to choose 2 from.

Simplifying this fraction we get $\frac{(30-k)(29-k)}{29\times 28}$. When this fraction is greater than 1/3, then staying is the best option. (Remember from class, the chance of winning when you switch is $\frac{1-x}{2}$, where x

is the probability when staying with the original choice. Solving for $x > \frac{1-x}{2}$, which is what holds when staying is the best strategy, is true when $x > \frac{1}{3}$.

It follows that we aim to find the maximum integer value for k for which the probability above is greater than 1/3. (It would also be fine to set it greater than or equal to 1/3.)

$$\frac{(30-k)(29-k)}{29 \times 28} > \frac{1}{3}$$

$$870 - 59k + k^2 > \frac{812}{3}$$

$$3k^2 - 177k + 2610 - 812 > 0$$

$$3k^2 - 177k + 1798 > 0$$

Let's find when the function on the left is equal to 0 via the quadratic formula:

$$k = \frac{177 \pm \sqrt{177^2 - 4(3)(1798)}}{6}$$

$$k = \frac{177 \pm \sqrt{9753}}{6}$$

Notice that the larger root is when k > 30. It follows that the value we care about is the smaller root, when k < 30. The function for negative k, is positive, then it drops to be negative, and then becomes positive again sometime after k > 30. Thus, the answer to the posed question is $\left\lfloor \frac{177-\sqrt{9753}}{6} \right\rfloor = \mathbf{13}$.

Let's plug in to ensure this is correct. If you choose 13 on your first turn and you stay, your chance of winning is $\frac{17\times16}{29\times28} = \frac{272}{812} = \frac{68}{203} > \frac{1}{3}$, because 3 x 68 = 204.

Alternatively, if you choose 14 on your first turn and stay, your chance of winning is $\frac{16\times15}{29\times28} = \frac{240}{812} = \frac{60}{203} < \frac{1}{3}$, because 3 x 60 = 180.

3) (6 pts) Consider the following discrete random variable X:

X = 2, with probability 0.1
= 8, with probability 0.2
= 12, with probability 0.5
= 16, with probability 0.2

Calculate both E(X) and Var(X).

Solution

$$E(X) = .1 \times 2 + .2 \times 8 + .5 \times 12 + .2 \times 16 = \mathbf{11}$$

$$E(X^2) = .1 \times 2^2 + .2 \times 8^2 + .5 \times 12^2 + .2 \times 16^2 = 136.4$$

$$Var(X) = E(X^2) - [E(X)]^2 = 136.4 - 11^2 = \mathbf{15.4}$$

4) (5 pts) Sam's probability of getting an A on an individual test is 75%. If he takes 12 tests, what is the probability he gets As on exactly 10 of those tests?

Solution

Using the binomial distribution, the probability Sam gets As on exactly 10 of those tests is

$$\boldsymbol{\binom{12}{10}(.75)^{10}(.25)^{2}} = \frac{66 \times 3^{10}}{4^{12}} = \frac{11 \times 3^{11}}{2^{23}} = \boldsymbol{\frac{1948617}{8388608} \sim 0.2323}$$

5) (10 pts) Here are the rules for Arup's Game of Dice:

1) Roll a pair of dice.
2) If you roll a sum of 11 or 12, you win.
3) If you roll a sum of 2, you lose.
4) Otherwise, record what you've rolled. Let this sum be k; also known as your point.
5) Roll one more time. If this roll exceeds your point(k), you win!
6) If this roll is the same as your point(k) or lower, you lose.

Calculate the probability that a player wins Arup's Game of Dice.

Solution

For a value k, we must determine the probability of rolling a sum greater than k on a pair of dice. First, let's fill in the chart for rolling each individual sum from 2 to 12, inclusive:

sum	2	3	4	5	6	7	8	9	10	11	12
prob	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Now, transform this table so that each entry represents the probability of rolling greater than that value. This simply represents the sums of the probabilities strictly to the right of that entry in the table above. This yields the following table:

sum	2	3	4	5	6	7	8	9	10	11	12
pbeat	35/36	33/36	30/36	26/36	21/36	15/36	10/36	6/36	3/36	1/36	0

Now, we are ready to write out the probability of winning Arup's Game of Dice. Here are the following, mutually exclusive cases for winning:

1) Rolling 11
2) Rolling 12
3) Rolling 3,4,5,6,7,8,9,10, followed by a larger sum on a pair.

Let's add these probabilities:

$$\frac{2}{36}+\frac{1}{36}+\frac{2}{36}\times\frac{33}{36}+\frac{3}{36}\times\frac{30}{36}+\frac{4}{36}\times\frac{26}{36}+\frac{5}{36}\times\frac{21}{36}+\frac{6}{36}\times\frac{15}{36}+\frac{5}{36}\times\frac{10}{36}+\frac{4}{36}\times\frac{6}{36}+\frac{3}{36}\times\frac{3}{36}=$$

$$\frac{72+36+66+90+104+105+90+50+24+9}{36\times 36}=\frac{646}{1296}=\mathbf{\frac{323}{648}}$$

Notice that $\frac{1}{2}=\frac{624}{648}$, so Arup is a pretty fair guy…but eventually, over the long haul, he'll take your money, with a $\frac{1}{648}$ advantage =)

Random note: I created this game for COP 3223 many years ago to be more "fair" than craps.

6) (6 pts) Define a continuous random variable as follows:

$f(x) = \frac{1}{9}x^2$, for $0 \le x \le 3$, amd
= 0, otherwise

Determine both the expectation and variance of the continuous random variable defined above.

Solution

First, let's calculate both E(X) and E(X^2):

$$E(X) = \int_0^3 \frac{1}{9}x^2(x)dx = \frac{1}{9}\int_0^3 x^3 = \frac{1}{36}x^4|_0^3 = \frac{81}{36} = \mathbf{\frac{9}{4}}$$

$$E(X^2) = \int_0^3 \frac{1}{9}x^2(x^2)dx = \frac{1}{9}\int_0^3 x^4 = \frac{1}{45}x^5|_0^3 = \frac{243}{45} = \frac{27}{5}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{27}{5} - \left(\frac{9}{4}\right)^2 = \frac{27}{5} - \frac{81}{16} = \frac{432 - 405}{80} = \mathbf{\frac{27}{80}}$$

7) (5 pts) Suppose E and F are events in a sample space and p(E) = 1/3, p(F) = 2/5, and p(F | E) = 1/2. Find p(E | F).

Solution

First, let's find p(E and F):

$$\frac{1}{2} = p(F|E) = \frac{p(F \cap E)}{p(E)} = \frac{p(E \cap F)}{\frac{1}{3}}$$

It follows that $p(E \cap F) = \frac{1/2}{3} = \frac{1}{6}$.

$$p(E|F) = \frac{p(E \cap F)}{p(F)} = \frac{\frac{1}{6}}{\frac{2}{5}} = \mathbf{\frac{5}{12}}$$

8) (5 pts) Give a summary of the life (thus far!) and mathematical contributions of Jennifer Balakrishnan.

Sample Summary

Jennifer Balakrishnan is an American mathematician, who currently serves as an associate professor of mathematics at Boston University. She was born in Guam and had an affinity towards mathematics. In her junior year of high school, she started studying elliptic coordinate systems and was awarded Honorable Mention in the 2001 Karl Menger Memorial Award, for her project for the Intel International Science and Engineering Fair. In addition, as a senior, she won the National High School Student Calculus Competition, which is given as part of the United States of America Mathematical Olympiad (USAMO).

Dr. Balakrishnan went on to do her undergraduate studies at Harvard and moved to MIT to complete her Ph.D. in mathematics under Kiran Kedlaya. Her dissertation was on "Coleman integration for hyperelliptic curves; algorithms and applications." After receiving her Ph.D. she did a post-doc at Harvard, followed by being a research fellow at Balliol College. In 2016, she became a faculty member at Boston University, where she currently is. She is known for her work in arithmetic geometry and number theory. In particular, she led a group of mathematicians in proving that the seven known rational solutions to the following curve:

$$y^4 + 5x^4 - 6x^2y^2 + 6x^3z + 26x^2yz + 10xy^2z - 10y^3z - 32x^2z^2 - 40xyx^2 + 24y^2z^2 + 32xz^3 - 16yz^3 = 0$$

are the only rational roots. In number theory, she has implemented many number theoretic algorithms for the SageMath computer algebra system.