

COT 3100 Fall 2022 Homework #8 Solutions

1) (10 pts) How many solutions does the equation $a + b + c + d + e = 45$ have, where a, b, c, d and e are each non-negative integers, under the following restrictions?

Solution

a) No other restrictions

This is exactly the combinations with repetition problem with $n = 45$ and $r = 5$. It follows that the number of solutions is $\binom{45 + 5 - 1}{5 - 1} = \binom{49}{4} = \binom{49}{45}$.

b) $d \geq 5$

Let $d = 5 + d'$, with integer $d' \geq 0$. Then we are looking for the number of non-negative solutions to the equation

$$\begin{aligned} a + b + c + d' + 5 + e &= 45 \\ a + b + c + d' + e &= 40 \end{aligned}$$

Using the combinations with repetition formula with $n = 40$, $r = 5$, we have $\binom{44}{4} = \binom{44}{40}$ possible solutions with this restriction.

c) $b \leq 10$

First, let's count the number of solutions with $b \geq 11$. This will represent all the solutions we *shouldn't* count. Let $b = 11 + b'$, with integer $b' \geq 0$. Substituting, we find that we want the number of non-negative integer solutions to the equation

$$\begin{aligned} a + 11 + b' + c + d + e &= 45 \\ a + b' + c + d + e &= 34 \end{aligned}$$

Using the combinations with repetition formula with $n = 34$, $r = 5$, we have $\binom{38}{4}$ possible solutions with $b \geq 11$.

It follows that the number of solutions to the original equation with $b \leq 10$ is

$$\binom{49}{4} - \binom{38}{4}$$

d) $b \leq 10$ and $c \leq 8$

Building off the solution to part c, we want to subtract out the number of solutions with $c \geq 9$. In doing so, we'll subtract out solutions with $b \geq 11$ AND $c \geq 9$ twice and then have to add those back in. Thus, we need to solve the two following subproblems:

1. Number of solutions with $c \geq 9$
2. Number of solutions with $c \geq 9$ AND $b \geq 11$.

For #1, let $c = 9 + c'$, where integer $c' \geq 0$. Then we want the number of solutions to $a + b + c' + d + e = 36$. There are $\binom{40}{4}$ possible solutions with $b \geq 11$.

For #2, let $b = 11 + b'$ and $c = 9 + c'$ for integers $b', c' \geq 0$. Then we want the number of solutions to $a + b' + c' + d + e = 25$. There are $\binom{29}{4}$ possible solutions with $b \geq 11$.

It follows that our final answer is $\binom{49}{4} - \binom{38}{4} - \binom{40}{4} + \binom{29}{4}$.

e) $d \geq 5$ and $b \leq 10$ and $c \leq 8$

In part (b), we took care of the restriction for $d \geq 5$. There are $\binom{44}{4}$ solutions to the equation with $d \geq 5$.

Next, of these solutions, we need to find how many have $b \geq 11$. Let $b = 11 + b'$, for an integer $b' \geq 0$. Then we are looking for the number of non-negative integer solutions to $a + b' + c + d + e = 29$. There are $\binom{33}{4}$ such solutions.

Similarly, we need to find how many of these solutions have $c \geq 9$. Let $c = 9 + c'$, for an integer $c' \geq 0$. Then we are looking for the number of non-negative integer solutions to $a + b + c' + d + e = 31$. There are $\binom{35}{4}$ such solutions.

Finally we need to find how many of the solutions we subtract out twice with $b \geq 11$ and $c \geq 9$. Let $b = 11 + b'$ and $c = 9 + c'$ for integers $b', c' \geq 0$. Then we are looking for the number of non-negative integer solutions to $a + b' + c' + d + e = 20$. There are $\binom{24}{4}$ such solutions.

It follows that the total number of solutions with the three given restrictions is

$$\binom{44}{4} - \binom{33}{4} - \binom{35}{4} + \binom{24}{4}$$

2) (5 pts) How many solutions does the equation $a + b + c + d + e + f + g + h + i + j \leq 55$ have if each variable must be a non-negative integer?

Solution

Add an extra variable, k , with $k \geq 0$, such that k is set to the difference between 55 and the sum $a + b + c + d + e + f + g + h + i + j$. Thus, there is a one-to-one correspondence between the non-negative integer solutions to the equation above and the number of non-negative integer solutions to the equation

$$a + b + c + d + e + f + g + h + i + j + k = 55$$

Using the combinations with repetition formula with $n = 55$, $r = 11$, there are $\binom{65}{10}$ such solutions.

3) (10 pts) How many ordered quadruplets (a_1, a_2, a_3, a_4) of non-negative integers, where at least one of the integers is even, satisfy the equation $a_1 + a_2 + a_3 + a_4 = 100$? Please express your answer in the form $\binom{w}{x} - \binom{y}{z}$. (Note that the values of w , x , y and z will be integers, but not necessarily all distinct.)

Solution

First, let's find the total number of odd solutions to the equation above, and then we can subtract that from the total number of solutions to get the ones with at least one even integer.

If all four integers are odd, there exist non-negative integers w , x , y and z such that $a_1 = 2w + 1$, $a_2 = 2x + 1$, $a_3 = 2y + 1$, and $a_4 = 2z + 1$. Thus, we are looking for the number of non-negative integer solutions to

$$\begin{aligned}(2w+1)+(2x+1)+(2y+1)+(2z+1) &= 100 \\ 2w+2x+2y+2z &= 96 \\ w+x+y+z &= 48\end{aligned}$$

Using the combinations with repetition formula with $n = 48$, $r = 4$, we find there are $\binom{51}{3}$ solutions to this equation, so there are $\binom{51}{3}$ solutions to the original equation in non-negative odd integers.

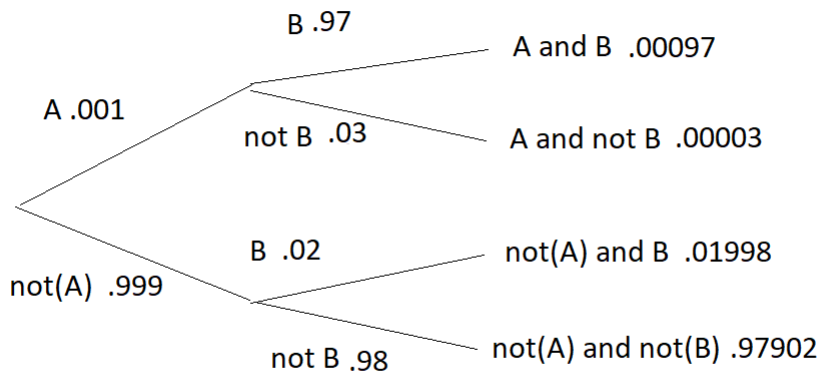
The total number of solutions to the original equation using the combinations with repetition formula with $n = 100$ and $r = 4$ is $\binom{103}{3}$. It follows that the total number of solutions to the equation where at least one integer is even is:

$$\binom{103}{3} - \binom{51}{3}$$

4) (5 pts) Suppose that one person in 1,000 people has a rare genetic disease. There is an excellent test for the disease; 97% of the people with the disease test positive and only 2% of the people who don't have it test positive. What is the probability that someone who tests positive has the disease? What is the probability that someone who tests negative does not have the disease?

Solution

Here is the tree diagram for the problem:



We aim to find $p(A | B)$. Note that we must find $p(B)$ first:

$$p(B) = p(A \cap B) + p(\bar{A} \cap B) = .00097 + .01998 = .02095$$

Now, we can answer the first question:

$$p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{.00097}{.02095} \sim 0.04630071599045346$$

Here is the answer to the second question:

$$p(\bar{A}|\bar{B}) = \frac{p(\bar{A} \cap \bar{B})}{p(\bar{B})} = \frac{.97902}{1 - .02095} = \frac{97902}{97905} \sim 0.999969358051172$$

5) (5 pts) A partial deck of Uno cards has 20 red cards and 20 blue cards. You draw, without replacement, 5 cards from the deck. The “most equal” split of colors you can receive is 2 cards of one color and three cards of the other color. What is the probability that you’ll receive this “most equal” split of colors?

Solution

The sample space is the number of ways to draw 5 cards out of 40, $\binom{40}{5}$.

Of these ways, we must have either 2 or 3 red cards. We can choose exactly 2 red cards and 3 blue cards in $\binom{20}{2} \binom{20}{3}$ ways, since we are choosing from both set sizes 20.

Similarly, we can choose 3 red cards and 2 blue cards in $\binom{20}{3} \binom{20}{2}$ ways.

It follows that the desired probability is $\frac{2\binom{20}{2}\binom{20}{3}}{\binom{40}{5}} = \frac{120 \times 20 \times 19 \times 20 \times 19 \times 18}{40 \times 39 \times 38 \times 37 \times 36 \times 6} = \frac{10 \times 5 \times 19}{13 \times 37 \times 3} = \frac{950}{1443} \sim .658$

6) (5 pts) The integers from 1 to 15, inclusive, are partitioned at random into two sets, one with 7 elements and the other with 8. What is the probability that 1 and 2 are in the same set?

Solution

We split our work into two cases: (a) 1 is in the set with 7 elements, (b) 1 is in the set with 8 elements. The probability of the former is 7/15 and the latter is 8/15.

Assuming that 1 is in the set with 7 elements, there are 6 more open slots in that set and 8 open slots in the other. Each slot is equally likely to be chosen for 2. It follows that in this case, there is a $\frac{6}{14}$ chance that 2 is in the same set as 1.

Assuming that 1 is in the set with 8 elements, there are 7 more open slots in that set and 7 open slots in the other. Each slot is equally likely to be chosen for 2. It follows that in this case, there is a $\frac{7}{14} = \frac{1}{2}$ chance that 2 is in the same set as 1.

Adding our probabilities from both branches we get: $\frac{7}{15} \times \frac{6}{14} + \frac{8}{15} \times \frac{7}{14} = \frac{7(6+8)}{15 \times 14} = \frac{7}{15}$

7) (5 pts) In a dice game, a player rolls 6 fair standard six sided dice. The player gets some points for all of the following:

- a) Three **or more** (of a kind, so three or more 4's for example)
- b) Three pairs (so two 3s, two 4s and two 6s for example)
- c) All six distinct numbers (so in any order 1, 2, 3, 4, 5, 6)
- d) Rolling the number 1 or the number 5

What is the probability, when rolling the six dice, that a player gets no points?

Solution

In order to get no points, the player must roll exactly 4 distinct values, since 1 and 5 score points and if the player only rolled 3 distinct values, they would either have 3 pairs or at least one three of a kind. Thus, the player must roll a 2, 3, 4 and 6 and must repeat exactly 2 of those rolls.

The sample space is 6^6 , the total number of ways the six dice could land.

We can think of all the ways of scoring no points as permutations of 2346xy, where x and y are distinct values from the set {2,3,4,6}. First, we choose these 2 distinct values in $\binom{4}{2} = 6$ ways.

For each of these choices, we can permute our string in $\frac{6!}{2!2!} = \frac{720}{4} = 180$ ways.

It follows that our desired probability is $\frac{6 \times 180}{6^6} = \frac{30}{6^4} = \frac{5}{216}$ ways.

8) (5 pts) Give a summary of the life and mathematical contributions of Evariste Galois. Please aim for a length of roughly 200 - 400 words. **Your summary must be typed.** Please state the sources you used in writing your summary.

Sample Summary

Galois was born in 1811 and his father was a prominent Republican, which came into play later in his life. Evariste Galois made contributions to nearly all branches of mathematics during his short life. For many times, his discoveries failed to be published due to various troubles.

He is one of the founders of the branch of algebra known as group theory. He developed the concept of a normal subgroup and Galois field. He made basic studies of linear groups over finite fields. He constructed the general linear group over a prime field and projective special linear group.

He developed Galois theory which is his most significant contribution to mathematics. He related the algebraic solution to a polynomial equation to the structure of a group of permutations associated with the roots of the polynomial. He found if an equation's Galois group has a series of subgroups, it could be solved in radicals.

For the analysis mathematics, Evariste made some contributions to the theory of Abelian integrals and continued fractions. He classified Abelian integrals into three categories.

Evariste also did important work on continued fractions. He proved that the regular continued fraction which represents a quadratic surd ζ is purely periodic if and only if ζ is a reduced surd. Evariste Galois's work had a significant overlap with the work of Niels Henrik Abel, another mathematician.

On a historical note, Galois was extremely interested in politics. In 1827, Charles X had suffered a political setback, but in 1830, he staged a coup d'etat. As this was occurring, Galois's school, the Ecole Normale was locked. Rather than deal with this, Galois quit school and joined the Republican artillery unit of the National Guard. Based on this association, Galois was put in jail (and eventually released) multiple times. Naturally, between being in jail and his various political activities, he has less time to work on mathematics. Nonetheless, he still made some contributions during this time. Unfortunately, Galois died in a duel that took place on May 30, 1832. The circumstances around the duel are quite mysterious, but there's some reasonable evidence that the duel was over a woman, but there's no credible account of who the killer was and the written trail of evidence is fairly scant. It's shocking that someone who died at such a young age (20 years old), was able to start a new branch of mathematics. Incidentally, much of modern public key cryptography works due to properties of Group Theory.

Sources: https://en.wikipedia.org/wiki/%C3%89variste_Galois